The precise boundary trace of positive solutions of the equation $\Delta u = u^q$ in the supercritical case.

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To Haïm, with friendship and high esteem.

ABSTRACT. We construct the precise boundary trace of positive solutions of $\Delta u = u^q$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, for q in the super-critical case $q \geq (N+1)/(N-1)$. The construction is performed in the framework of the fine topology associated with the Bessel capacity $C_{2/q,q'}$ on $\partial\Omega$. We prove that the boundary trace is a Borel measure (in general unbounded),which is outer regular and essentially absolutely continuous relative to this capacity. We provide a necessary and sufficient condition for such measures to be the boundary trace of a positive solution and prove that the corresponding generalized boundary value problem is well-posed in the class of σ-moderate solutions.

1. Introduction

In this paper we present a theory of boundary trace of positive solutions of the equation

$$-\Delta u + |u|^{q-1}u = 0 \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ of class C^2. A function u is a solution if $u \in L^q_{loc}(\Omega)$ and the equation holds in the distribution sense.

Semilinear elliptic equations with absorption, of which (1.1) is one of the most important, have been intensively studied in the last 30 years. The foundation for these studies can be found in the pioneering work of Brezis starting with his joint research with Benilan in the 70's [**2**], and followed by a series of works with colleagues and students, up to the present.

In the subcritical case, $1 < q < q_c = (N+1)/(N-1)$, the boundary trace theory and the associated boundary value problem, are well understood. This theory has been developed, in parallel, by two different methods: one based on a probabilistic approach (see [**11, 5, 6**], Dynkin's book [**3**] and the references therein) and the other purely analytic (see [**13, 14, 15**]). In 1997 Le Gall showed that this theory is not appropriate for the supercritical case because, in this case, there

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may be infinitely many solutions with the same boundary trace. Following this observation, a theory of 'fine' trace was introduced by Dynkin and Kuznetsov [**7**]. Their results demonstrated that, for $q \leq 2$, the fine trace theory is satisfactory in the family of so-called σ-moderate solutions. A few years later Mselati [**17**] used this theory and other results of Dynkin [**3**], in combination with the Brownian snake method developed by Le Gall [**12**], in order to show that, in the case $q = 2$ all positive solutions are σ-moderate. Shortly thereafter Marcus and Veron [**16**] proved that, for all $q \geq q_c$ and every compact set $K \subset \partial\Omega$, the maximal solution of (1.1) vanishing outside K is σ-moderate. Their proof was based on the derivation of sharp capacitary estimates for the maximal solution. In continuation, Dynkin [**4**] used Mselati's (probabilistic) approach and the results of Marcus and Veron [**16**] to show that, in the case $q \leq 2$, all positive solutions are σ-moderate. For $q > 2$ the problem remains open.

Our definition of boundary trace is based on the fine topology associated with the Bessel capacity $C_{2/q,q'}$ on $\partial\Omega$, denoted by $\mathfrak{T}_q$. The presentation requires some notation.

Notation 1.1.

a: For every $x \in \mathbb{R}^N$ and every $\beta > 0$ put $\rho(x) := \mathrm{dist}\,(x, \partial\Omega)$ and

$$\Omega_\beta = \{x \in \Omega : \, \rho(x) < \beta\}, \quad \Omega'_\beta = \Omega \setminus \bar{\Omega}_\beta, \quad \Sigma_\beta = \partial\Omega'_\beta.$$

b: There exists a positive number β_0 such that,

$$\forall x \in \Omega_{\beta_0} \quad \exists! \; \sigma(x) \in \partial\Omega \, : \, \mathrm{dist}\,(x, \sigma(x)) = \rho(x). \tag{1.2}$$

If (as we assume) Ω is of class C^2 and β_0 is sufficiently small, the mapping $x \mapsto (\rho(x), \sigma(x))$ is a C^2 diffeomorphism of Ω_{β_0} onto $(0, \beta_0) \times \partial\Omega$.

c: If $Q \subset \partial\Omega$ put $\Sigma_\beta(Q) = \{x \in \Sigma_\beta : \, \sigma(x) \in Q\}$.

d: If Q is a $\mathfrak{T}_q$-open subset of $\partial\Omega$ and $u \in C(\partial\Omega)$ we denote by u^Q_β the solution of (1.1) in Ω'_β with boundary data $h = u\chi_{_{\Sigma_\beta(A)}}$ on Σ_β.

Recall that a solution u is moderate if $|u|$ is dominated by a harmonic function. When this is the case, u possesses a boundary trace (denoted by $\mathrm{tr}\, u$) given by a bounded Borel measure. The boundary trace is attained in the sense of weak convergence, as in the case of positive harmonic functions (see [**13**] and the references therein). If $\mathrm{tr}\, u$ happens to be absolutely continuous relative to Hausdorff $(N-1)$-dimensional measure on $\partial\Omega$ we refer to its density f as the L^1 boundary trace of u and write $\mathrm{tr}\, u = f$ (which should be seen as an abbreviation for $\mathrm{tr}\, u = f d\,\mathbb{H}_{N-1}$).

A positive solution u is σ-moderate if there exists an increasing sequence of moderate solutions $\{u_n\}$ such that $u_n \uparrow u$. This notion was introduced by Dynkin and Kuznetsov [**7**] (see also [**9**] and [**3**]).

If μ is a bounded Borel measure on $\partial\Omega$, the problem

$$-\Delta u + u^q = 0 \text{ in } \Omega, \quad u = \mu \text{ on } \partial\Omega \tag{1.3}$$

possesses a (unique) solution if and only if μ vanishes on sets of $C_{2/q,q'}$-capacity zero, (see [**15**] and the references therein). The solution is denoted by u_μ.

The set of positive solutions of (1.1) in Ω will be denoted by $\mathcal{U}(\Omega)$. It is well known that this set is compact in the topology of $C(\Omega)$, i.e., relative to local uniform convergence in Ω.

Our first result displays a dichotomy which is the basis for our definition of boundary trace.

THEOREM 1.1. *Let u be a positive solution of* (1.1) *and let $\xi \in \partial\Omega$. Then,* **either,** *for every $\mathfrak{T}_q$-open neighborhood Q of ξ, we have*

$$\lim_{\beta\to 0}\int_{\Sigma_\beta(Q)} udS = \infty \tag{1.4}$$

or *there exists a $\mathfrak{T}_q$-open neighborhood Q of ξ such that*

$$\lim_{\beta\to 0}\int_{\Sigma_\beta(Q)} udS < \infty. \tag{1.5}$$

The first case occurs if and only if

$$\int_D u^q\rho(x)dx = \infty, \quad D = (0,\beta_0)\times Q \tag{1.6}$$

for every $\mathfrak{T}_q$-open neighborhood Q of ξ.

A point $\xi \in \partial\Omega$ is called a *singular* point of u in the first case, i.e. when (1.4) holds, and a *regular* point of u in the second case. The set of singular points is denoted by $\mathcal{S}(u)$ and its complement in $\partial\Omega$ by $\mathcal{R}(u)$.

Our next result provides additional information on the behavior of solutions near the regular boundary set $\mathcal{R}(u)$.

THEOREM 1.2. *The set of regular points $\mathcal{R}(u)$ is $\mathfrak{T}_q$-open and there exists a non-negative Borel measure μ on $\partial\Omega$ possessing the following properties.*
(i) For every $\sigma \in \mathcal{R}(u)$ there exist a $\mathfrak{T}_q$-open neighborhood Q of σ and a moderate solution w such that

$$\tilde{Q} \subset \mathcal{R}(u), \quad \mu(\tilde{Q}) < \infty, \tag{1.7}$$

and

$$u_\beta^Q \to w \ \text{ locally uniformly in } \Omega \quad , \quad (\mathrm{tr}\, w)\chi_Q = \mu\chi_Q. \tag{1.8}$$

(ii) μ is outer regular relative to $\mathfrak{T}_q$.

Based on these results we define the *precise boundary trace* of u by

$$\mathrm{tr}^{\,c} u = (\mu, \mathcal{S}(u)). \tag{1.9}$$

Thus a trace is represented by a couple $(\mu, \mathcal{S})$, where $\mathcal{S} \subset \partial\Omega$ is $\mathfrak{T}_q$-closed and μ is an outer regular measure relative to $\mathfrak{T}_q$ which is $\mathfrak{T}_q$-locally finite on $\mathcal{R} = \partial\Omega\setminus\mathcal{S}$. However, not every couple of this type is a trace. A necessary and sufficient condition for such a couple to be a trace is provided in Theorem 5.16.

The trace can also be represented by a Borel measure ν defined as follows:

$$\nu(A) = \begin{cases} \mu(A) & \text{if } A \subset \mathcal{R}(u), \\ \infty & \text{otherwise,} \end{cases} \tag{1.10}$$

for every Borel set $A \subset \partial\Omega$. We put

$$\mathrm{tr}\, u := \nu. \tag{1.11}$$

This measure has the following properties:
(i) It is outer regular relative to $\mathfrak{T}_q$.
(ii) It is *essentially absolutely continuous* relative to $C_{2/q,q'}$, i.e., for every $\mathfrak{T}_q$-open set Q and every Borel set A such that $C_{2/q,q'}(A) = 0$,

$$\nu(Q) = \nu(Q \setminus A).$$

The second property will be denoted by $\nu \underset{f}{\prec\prec} C_{2/q,q'}$. It implies that, if $\nu(Q\setminus A)<\infty$ then $\nu(Q\cap A)=0$. In particular, ν is absolutely continuous relative to $C_{2/q,q'}$ on $\mathfrak{T}_q$-open sets on which it is bounded.

A positive Borel measure possessing properties (i) and (ii) will be called a *q-perfect measure*. The space of q-perfect measures will be denoted by $\mathbb{M}_q(\partial\Omega)$.

We have the following necessary and sufficient condition for existence:

THEOREM 1.3. *Let ν be a positive Borel measure on $\partial\Omega$, possibly unbounded. The boundary value problem*

$$-\Delta u+u^q=0,\quad u>0 \text{ in } \Omega,\quad \mathrm{tr}\,(u)=\nu \text{ on } \partial\Omega \tag{1.12}$$

possesses a solution if and only if ν is q-perfect. When this condition holds, a solution of (1.12) *is given by*

$$U=v\oplus U_F,\quad v=\sup\{u_{\nu\chi_Q}:Q\in\mathcal{F}_\nu\}, \tag{1.13}$$

where

$$\mathcal{F}_\nu:=\{Q:\ Q \text{ q-open},\ \nu(Q)<\infty\},\quad G:=\bigcup_{\mathcal{F}_n u}Q,\quad F=\partial\Omega\setminus G$$

and U_F is the maximal solution vanishing on $\partial\Omega\setminus F$.

Finally we establish the following uniqueness result.

THEOREM 1.4. *Let ν be a q-perfect measure on $\partial\Omega$. Then the solution U of problem* (1.12) *defined by* (1.13) *is σ-moderate and it is the maximal solution with boundary trace ν. Furthermore, the solution of* (1.12) *is unique in the class of σ-moderate solutions.*

For $q_c\leq q\leq 2$, results similar to those stated in the last two theorems, were obtained by Dynkin and Kuznetsov [**7**] and Kuznetsov [**9**], based on their definition of fine trace. However, by their results, the prescribed trace is attained only up to equivalence, i.e., up to a set of capacity zero. By the present results, the solution attains precisely the prescribed trace and this holds for all values of q in the supercritical range. The relation between the Dynkin-Kuznetsov definition (which is used in a probabilistic formulation) and the definition presented here, is not yet clear.

The plan of the paper is as follows:

Section 2 presents results on the $C_{2/q,q'}$-fine topology which, for brevity, is called the q-topology.

Section 3 deals with the concept of maximal solutions which vanish on the boundary outside a q-closed set. Included here is a sharp estimate for these solutions, based on the capacitary estimates developed by the authors in [**16**]. In particular we prove that the maximal solutions are σ-moderate. This was established in [**16**] for solutions vanishing on the boundary outside a *compact set*.

Section 4 is devoted to the problem of localization of solutions in terms of boundary behavior. Localization methods are of crucial importance in the study of trace and the associated boundary value problems. The development of these methods is particularly subtle in the supercritical case.

Section 5 presents the concept of precise trace and studies it, firstly on the regular boundary set, secondly in the case of σ-moderate solutions and finally in the general case. This section contains the proofs of the theorems stated above: Theorem 1.1

is a consequence of Theorem 5.7. Theorem 1.2 is a consequence of Theorem 5.11. Theorem 1.3 is a consequence of Theorem 5.16 (see the remark following the proof of the latter theorem). Finally Theorem 1.4 is contained in Theorem 5.16.

2. The q-fine topology

A basic ingredient in our study is the fine topology associated with a Bessel capacity on $(N-1)$-dimensional smooth manifolds. The theory of fine topology associated with the Bessel capacity $C_{\alpha,p}$ in $\mathbb{R}^N$ essentially requires $0<\alpha p\leq N$ (see [**1**, Chapter 6]). In this paper we are interested in the fine topology associated with the capacity $C_{2/q,q'}$ in $\mathbb{R}^{N-1}$ or on the boundary manifold $\partial\Omega$ of a smooth bounded domain $\Omega\in\mathbb{R}^N$. We assume that q is in the supercritical range for (1.1), i.e., $q\geq q_c=(N+1)/(N-1)$. Thus $2q'/q=2/(q-1)\leq N-1$. We shall refer to the $(2/q,q')$-fine topology briefly as the q-topology.

An important concept related to this topology is the $(2/q,q')$-quasi topology. We shall refer to it as the q-quasi topology. For definition and details see [**1**, Section 6.1-4].

We say that a subset of $\partial\Omega$ is q-open (resp. q-closed) if it is open (resp. closed) in the q-topology on $\partial\Omega$. The terms q-quasi open and q-quasi closed are understood in an analogous manner.

Notation 2.1. Let A,B be subsets of $\mathbb{R}^{N-1}$ or of $\partial\Omega$.

a: A is q-essentially contained in B, denoted $A\overset{q}{\subset}B$, if
$$C_{2/q,q'}(A\setminus B)=0.$$
b: The sets A,B are q-equivalent, denoted $A\overset{q}{\sim}B$, if
$$C_{2/q,q'}(A\Delta B)=0.$$
c: The q-fine closure of a set A is denoted by $\tilde{A}$. The q-fine interior of A is denoted by $A^\diamond$.

d: Given $\epsilon>0$, A^ϵ denotes the intersection of $\mathbb{R}^{N-1}$ (or $\partial\Omega$) with the ϵ-neighborhood of A in $\mathbb{R}^N$.

e: The set of $(2/q,q')$-thick (or briefly q-thick) points of A is denoted by $b_q(A)$. The set of $(2/q,q')$-thin (or briefly q-thin) points of A is denoted by $e_q(A)$, (for definition see [**1**, Def. 6.3.7]).

Remark. If $A\subset\partial\Omega$ and $B:=\partial\Omega\setminus A$ then
$$A \text{ is } q\text{-open} \iff A\subset e_q(B),\quad B \text{ is } q\text{-closed} \iff b_q(B)\subset B. \tag{2.1}$$
Consequently
$$\tilde{A}=A\cup b_q(A),\quad A^\diamond=A\cap e_q(\partial\Omega\setminus A), \tag{2.2}$$
(see [**1**, Section 6.4].)

The capacity $C_{2/q,q'}$ possesses the Kellogg property, namely,
$$C_{2/q,q'}(A\setminus b_q(A))=0, \tag{2.3}$$
(see [**1**, Cor. 6.3.17]). Therefore
$$A\overset{q}{\subset}b_q(A)\overset{q}{\sim}\tilde{A} \tag{2.4}$$
but, in general, $b_q(A)$ does not contain A. The Kellog property and (2.1) implies:

PROPOSITION 2.1. *(i) If Q is a q-open set then $\check{Q} := e_q(\partial\Omega \setminus Q)$ is the largest q-open set that is q-equivalent to Q.*
(ii) If F is a q-closed set then $\hat{F} = b_q(F)$ is the smallest q-closed set that is q-equivalent to F.

We collect below several facts concerning the q-fine topology that are used throughout the paper.

PROPOSITION 2.2. *Let $q_c = (N+1)/(N-1) \leq q$.*

i: *Every q-closed set is q-quasi closed* [**1**, Prop. 6.4.13].
ii: *If E is q-quasi closed then $\tilde{E} \overset{q}{\sim} E$* [**1**, Prop. 6.4.12].
iii: *A set E is q-quasi closed if and only if there exists a sequence $\{E_m\}$ of compact subsets of E such that $C_{2/q,q'}(E \setminus E_m) \to 0$* [**1**, Prop. 6.4.9].
iv: *There exists a constant c such that, for every set E,*

$$C_{2/q,q'}(\tilde{E}) \leq cC_{2/q,q'}(E),$$

see [**1**, Prop. 6.4.11].
v: *If E is q-quasi closed and $F \overset{q}{\sim} E$ then F is q-quasi closed.*
vi: *If $\{E_i\}$ is an increasing sequence of arbitrary sets then*

$$C_{2/q,q'}(\cup E_i) = \lim_{i\to\infty} C_{2/q,q'}(E_i).$$

vii: *If $\{K_i\}$ is a decreasing sequence of compact sets then*

$$C_{2/q,q'}(\cap K_i) = \lim_{i\to\infty} C_{2/q,q'}(K_i).$$

viii: *Every Suslin set and, in particular, every Borel set E satisfies*

$$\begin{aligned} C_{2/q,q'}(E) &= \sup\{C_{2/q,q'}(K) : K \subset E,\ K \text{ compact}\} \\ &= \inf\{C_{2/q,q'}(G) : E \subset G,\ G \text{ open}\}. \end{aligned}$$

For the last three statements see [**1**, Sec. 2.3]. Statement (v) is an easy consequence of [**1**, Prop. 6.4.9]. However note that this assertion is no longer valid if 'q-quasi closed' is replaced by 'q-closed'. Only the following weaker statement holds:

If E is q-closed and A is a set such that $C_{2/q,q'}(A) = 0$ then $E \cup A$ is q-closed.

DEFINITION 2.3. Let E be a quasi closed set. An increasing sequence $\{E_m\}$ of compact subsets of E such that $C_{2/q,q'}(E \setminus E_m) \to 0$ is called a *q-stratification of E*.
(i) We say that $\{E_m\}$ is a *proper q-stratification of E* if

$$C_{2/q,q'}(E_{m+1} \setminus E_m) \leq 2^{-m-1} C_{2/q,q'}(E). \tag{2.5}$$

(ii) Let $\{\epsilon_m\}$ be a strictly decreasing sequence of positive numbers converging to zero such that

$$C_{2/q,q'}(G_{m+1} \setminus G_m) \leq 2^{-m} C_{2/q,q'}(E), \quad G_m := \cup_{k=1}^m E_k^{\epsilon_k}. \tag{2.6}$$

The sequence $\{\epsilon_m\}$ is called a *q-proper sequence.*
(iii) If V is a q-open set such that $C_{2/q,q'}(E \setminus V) = 0$ we say that V is a *q-quasi neighborhood* of E.

Remark. Observe that $G := \cup_{k=1}^\infty E_k^{\epsilon_k}$ is a q-open neighborhood of $E' = \cup E_m$ but, in general, only a q-quasi neighborhood of E.

Lemma 2.4. *Let E be a q-closed set such that $C_{2/q,q'}(E)>0$. Then:*
(i) Let D be an open set such that $C_{2/q,q'}(E\setminus D)=0$. Then $E\cap D$ is q-quasi closed and consequently there exists a proper q-stratification of $E\cap D$, say $\{E_m\}$. Furthermore, there exists a q-proper sequence $\{\epsilon_m\}$ such that

$$G=\cup_{m=1}^{\infty}(E_m)^{\epsilon_m}\subset D$$

and

$$\cup E_m=E'\subset O\subset\tilde{O}\overset{q}{\subset} D \quad \text{where} \quad O:=\cup_{m=1}^{\infty}(E_m)^{\epsilon_m/2}. \tag{2.7}$$

Consequently

$$E\overset{q}{\subset} O\subset\tilde{O}\overset{q}{\subset} D. \tag{2.8}$$

(ii) If D is a q-open set such that $E\overset{q}{\subset} D$ then there exists a q-open set O such that (2.8) *holds.*

Proof. If A_1, A_2 are two sets such that $A_1\overset{q}{\sim}A_2$ and A_1 is q-quasi closed then A_2 is q-quasi closed, (see the discussion of the quasi topology in [**1**, sec. 6.4]). Since $E\cap D\overset{q}{\sim}E$ and E is q-closed it follows that $E\cap D$ is q-quasi closed. Let $\{E_m\}$ be a proper q-stratification of $E\cap D$ and put $E'=\cup_{m=1}^{\infty}E_m$. If E' is a closed set the remaining part of assertion (i) is trivial. Therefore we assume that E' is not closed and that

$$C_{2/q,q'}(E_{m+1}\setminus E_m)>0.$$

To prove the first statement we construct the sequence $\{\epsilon_m\}_{m=1}^{\infty}$ inductively so that (with $E_0=\emptyset$ and $\epsilon_0=1$) the following conditions are satisfied:

$$F_m:=E_m\setminus(E_{m-1})^{\frac{1}{2}\epsilon_{m-1}}, \qquad C_{2/q,q'}(F_m)>0, \tag{2.9}$$

$$C_{2/q,q'}(\overline{F_m^{\epsilon_m}})\leq 2C_{2/q,q'}(E_m\setminus E_{m-1}), \qquad \overline{F_m^{\epsilon_m}}\subset D, \tag{2.10}$$

$$\epsilon_m<\epsilon_{m-1}/2, \qquad m=1,2,\dots. \tag{2.11}$$

Choose $0<\epsilon_1<1/2$, sufficiently small so that

$$E_1^{\epsilon_1}\subset D, \quad C_{2/q,q'}(E_2\setminus E_1^{\epsilon_1/2})>0.$$

This is possible because our assumption implies that there exists a compact subset of $E_2\setminus E_1$ of positive capacity. By induction we obtain

$$E_m^{\epsilon_m}\subset E_{m-1}^{\epsilon_{m-1}}\cup F_m^{\epsilon_m}\subset D \tag{2.12}$$

and consequently

$$E_m^{\epsilon_m}\subset\cup_{k=1}^{m}F_k^{\epsilon_k}, \quad m=1,2,\dots. \tag{2.13}$$

Since $F_m\subset E_m$, (2.13) implies that

$$G:=\cup_{k=1}^{\infty}E_k^{\epsilon_k}=\cup_{k=1}^{\infty}F_k^{\epsilon_k}, \quad G_m:=\cup_{k=1}^{m}E_k^{\epsilon_k}=\cup_{k=1}^{m}F_k^{\epsilon_k}. \tag{2.14}$$

The sequence $\{\epsilon_m\}$ constructed above satisfies 2.6. Indeed, by (2.5), (2.10) and (2.14),

$$\begin{aligned} C_{2/q,q'}(G\setminus G_m) &\leq \sum_{k=m+1}^{\infty}C_{2/q,q'}(F_k^{\epsilon_k}) \\ &\leq 2\sum_{k=m+1}^{\infty}C_{2/q,q'}(E_k\setminus E_{k-1})\leq 2^{-m+1}C_{2/q,q'}(E). \end{aligned} \tag{2.15}$$

Next we show that the set

$$O' := \cup_{k=1}^{\infty}\overline{E_k^{\epsilon_k/2}}$$

is q-quasiclosed. By (2.13),

$$O' = \cup_{k=1}^{\infty}\overline{F_k^{\epsilon_k/2}}, \quad O'_m := \cup_{k=1}^{m}\overline{E_k^{\epsilon_k/2}} = \cup_{k=1}^{m}\overline{F_k^{\epsilon_k/2}}. \tag{2.16}$$

Hence, by (2.5) and (2.10),

$$C_{2/q,q'}(O' \setminus O'_m) \leq \sum_{k=m+1}^{\infty} C_{2/q,q'}(\overline{F_k^{\epsilon_k/2}}) \leq 2^{-m+1}C_{2/q,q'}(E). \tag{2.17}$$

Since O'_m is closed this implies that O' is quasiclosed. Further any quasiclosed set is equivalent to its fine closure. Since $O \subset O'$ it follows that $\tilde{O} \subset \tilde{O}' \overset{q}{\sim} O' \subset G$.

We turn to the proof of (ii) for which we need the following:

Assertion 1. Let D be a q-open set. Then there exists a sequence of relatively open sets $\{A_n\}$ such that

$$D_n := D \cup A_n \text{ is open}, \quad C_{2/q,q'}(\tilde{A}_n) \leq 2^{-n}, \quad \tilde{A}_{n+1} \overset{q}{\subset} A_n. \tag{2.18}$$

The sequence is constructed inductively. Let D'_1 be an open set such that $D \subset D'_1$ and $A'_1 = D'_1 \setminus D$ satisfies $C_{2/q,q'}(\tilde{A}'_1) < 1/4$. Let A_1 be an open set such that $\tilde{A}'_1 \subset A_1$ and $C_{2/q,q'}(\tilde{A}_1) \leq 1/2$. Assume that we constructed $\{A'_k\}_1^{n-1}$ and $\{A_k\}_1^{n-1}$ so that the sets A_k are open, (2.18) holds and

$$\begin{aligned} &D'_k = D \cup A'_k \text{ is open}, \quad \{D'_k\}_1^{n-1} \text{ is decreasing}, \\ &C_{2/q,q'}(\tilde{A}'_k) < 2^{-(k+1)}, \quad \tilde{A}'_k \subset A_k. \end{aligned} \tag{2.19}$$

Let D'_n be an open set such that

$$D \subset D'_n \subset D'_{n-1}, \quad C_{2/q,q'}(\tilde{A}'_n) < 2^{-(n+1)} \text{ where } A'_n = D'_n \setminus D.$$

Then $A'_n \subset A'_{n-1}$ and consequently $\tilde{A}'_n \subset A_{n-1}$. Since A_{n-1} is open, statement (i) implies that there exists an open set A_n such that

$$\tilde{A}'_n \overset{q}{\subset} A_n \subset \tilde{A}_n \overset{q}{\subset} A_{n-1}, \quad C_{2/q,q'}(\tilde{A}_n) \leq 2^{-n}.$$

This completes the proof of the assertion.

Let A_n and D_n be as in (2.18). By (i) there exists a q-open set Q such that

$$E \overset{q}{\subset} Q \subset \tilde{Q} \overset{q}{\subset} D_1.$$

Put

$$Q_n := Q \setminus (\cup_1^{n-1}(\tilde{A}_k \setminus A_{k+1}), \quad Q_\infty = Q \setminus (\cup_1^{\infty}(\tilde{A}_k \setminus A_{k+1}).$$

Then Q_n is a q-open set and we claim that

- **a:** Q_∞ is quasi open,
- **b:** $E \overset{q}{\subset} Q_n \overset{q}{\subset} D_n$,
- **c:** $\tilde{Q}_n \overset{q}{\subset} D_n \cup (\cup_1^n \partial_q A_i)$

Since Q_n is q-open, **a** follows from (2.18) which implies:

$$Q_\infty \cup (\cup_n^{\infty}(\tilde{A}_k \setminus A_{k+1}) = Q_n, \quad C_{2/q,q'}((\cup_n^{\infty}(\tilde{A}_k \setminus A_{k+1}) \leq 2^{n-1}.$$

We verify **b, c** by induction. Put $Q_1 = Q$ so that **b, c** hold for $n = 1$. If **b** holds for $n = 1, \cdots, j$ then,

$$Q_{j+1} = Q_j \setminus (\tilde{A}_j \setminus A_{j+1}), \quad E \overset{q}{\subset} Q_j \overset{q}{\subset} D_j,$$

which implies **b** for $n = j + 1$. If **c** holds for $n = 1, \cdots, j$ then,

$$\tilde{Q}_{j+1} \subset \tilde{Q}_j \setminus (A_j \setminus \tilde{A}_{j+1}) \overset{q}{\subset} D \cup (\cup_1^j \partial_q A_i) \cup \tilde{A}_{j+1}) = D_{j+1} \cup (\cup_1^{j+1} \partial_q A_i),$$

so that **c** holds for $n = j + 1$.

Taking the limit in **b** as $n \to \infty$ we obtain

$$E \overset{q}{\subset} Q_\infty \overset{q}{\subset} D.$$

Taking the limit in **c** as $n \to \infty$ we obtain

$$\tilde{Q}_\infty \overset{q}{\subset} D \cup (\cup_1^\infty \partial_q A_i).$$

However, by the same token,

$$\tilde{Q}_\infty \overset{q}{\subset} D \cup (\cup_k^\infty \partial_q A_i) \quad \forall k \in \mathbb{N}.$$

Therefore, by (2.18), $\tilde{Q}_\infty \overset{q}{\subset} D$. Thus (ii) holds with $O = Q_\infty$. □

Lemma 2.5. *Let E be a q-closed set and let $\mathcal{D}$ be a cover of E consisting of q-open sets. Then, for every $\epsilon > 0$ there exists an open set O_ϵ such that $C_{2/q,q'}(O_\epsilon) < \epsilon$ and $E \setminus O_\epsilon$ is covered by a finite subfamily of $\mathcal{D}$.*

Proof. By [**1**, Sec. 6.5.11] the (α, p)-fine topology possesses the quasi Lindelöf property. Thus there exists a denumerable subfamily of $\mathcal{D}$, say $\{D_n\}$, such that

$$O = \cup\{D : D \in \mathcal{D}\} \overset{q}{\sim} \cup D_n.$$

Let O_n be an open set containing D_n such that $C_{2/q,q'}(O_n \setminus D_n) < \epsilon/(2^n 3)$. Let K be a compact subset of $E \cap (\cup_1^\infty D_n)$ such that $C_{2/q,q'}(E \setminus K) < \epsilon/3$. Then $\{O_n\}$ is an open cover of K so that there exists a finite subcover of K, say $\{O_1, \cdots, O_k\}$. It follows that

$$C_{2/q,q'}(E \setminus \cup_{n=1}^k D_n) \leq C_{2/q,q'}(E \setminus K) + \sum_n C_{2/q,q'}(O_n \setminus D_n) < 2\epsilon/3.$$

Let O_ϵ be an open subset of $\partial\Omega$ such that $E \setminus \cup_{n=1}^k D_n \subset O_\epsilon$ and $C_{2/q,q'}(O_\epsilon) < \epsilon$. This set has the properties stated in the lemma. □

Lemma 2.6. *(a) Let E be a q-quasi closed set and $\{E_m\}$ a proper q-stratification for E. Then there exists a* decreasing *sequence of open sets $\{Q_j\}$ such that $\cup E_m := E' \subset Q_j$ for every $j \in \mathbb{N}$ and*

$$\begin{aligned} &(i) \quad \cap Q_j = E', \quad \tilde{Q}_{j+1} \overset{q}{\subset} Q_j, \\ &(ii) \quad \lim C_{2/q,q'}(Q_j) = C_{2/q,q'}(E). \end{aligned} \tag{2.20}$$

(b) If A is a q-open set, there exists a decreasing sequence of open sets $\{A_m\}$ such that

$$A \subset \cap A_m =: A', \qquad C_{2/q,q'}(A_m \setminus A') \to 0, \qquad A \overset{q}{\sim} A'. \tag{2.21}$$

Furthermore there exists an increasing *sequence of closed sets* $\{F_j\}$ *such that* $F_j \subset A'$ *and*

$$\begin{aligned}(i) &\quad \cup F_j = A', \quad F_j \overset{q}{\subset} F^{\diamond}_{j+1} =: D_{j+1}, \\ (ii) &\quad C_{2/q,q'}(F_j) \to C_{2/q,q'}(A) \qquad .\end{aligned} \tag{2.22}$$

Proof. (a) Let $\{\epsilon_m\}$ be a sequence of positive numbers decreasing to zero satisfying (2.6). Put

$$Q_j := \cup_{m=1}^{\infty} (E_m)^{\epsilon_m/2^j}.$$

Then $E' = \cap Q_j$ and

$$\tilde{Q}_j \overset{q}{\subset} Q'_j := \cup_{m=1}^{\infty} \overline{E_m^{\epsilon_m/2^j}} \subset Q_{j-1}. \tag{2.23}$$

Indeed Q'_j is quasi closed so that $\tilde{Q}'_j \overset{q}{\sim} Q'_j$. This proves (2.20)(i).

If D is a neighborhood of E' then, for every k there exists j_k such that

$$\cup_{m=1}^{k} (E_m)^{\epsilon_m/2^j} \subset D \quad \forall j \geq j_k.$$

Therefore,

$$C_{2/q,q'}(Q_j \setminus D) \leq C_{2/q,q'}(Q_j \setminus \cup_{m=1}^{k} (E_m)^{\epsilon_m/2^j}) \leq 2^{-k+1} C_{2/q,q'}(E) \quad \forall j \geq j_k.$$

Hence

$$C_{2/q,q'}(Q_j \setminus D) \to 0 \text{ as } j \to \infty. \tag{2.24}$$

Let $\{D_i\}$ be a decreasing sequence of open neighborhoods of E' such that

$$C_{2/q,q'}(D_i) \to C_{2/q,q'}(E').$$

By (2.24), for every i there exists $j(i) > i$ such that

$$C_{2/q,q'}(Q_{j(i)} \setminus D_i) \to 0 \text{ as } i \to \infty. \tag{2.25}$$

It follows that

$$C_{2/q,q'}(E') \leq \lim C_{2/q,q'}(Q_{j(i)}) \leq \lim C_{2/q,q'}(D_i) = C_{2/q,q'}(E') = C_{2/q,q'}(E).$$

This proves (2.20) (ii).

(b) Put $E = \partial\Omega \setminus A$ and let $\{E_m\}$ and $\{\epsilon_m\}$ be as in (a). Then (2.21) holds with $A_m := \partial\Omega \setminus E_m$. In addition, (2.22)(i) with $F_j := \partial\Omega \setminus Q_j$ is a consequence of (2.20)(i).

To verify (2.22) (ii) we observe that, if K is a compact subset of A' then, by (2.24),

$$C_{2/q,q'}(K \setminus F_j) \to 0.$$

Let $\{K_i\}$ be an increasing sequence of compact subsets of A' such that

$$C_{2/q,q'}(K_i) \uparrow C_{2/q,q'}(A') = C_{2/q,q'}(A).$$

As in part (a), for every i there exists $j(i) > i$ such that

$$C_{2/q,q'}(K_i \setminus F_{j(i)}) \to 0 \text{ as } i \to \infty. \tag{2.26}$$

It follows that

$$C_{2/q,q'}(A') \geq \lim C_{2/q,q'}(F_{j(i)}) \geq \lim C_{2/q,q'}(K_i) = C_{2/q,q'}(A') = C_{2/q,q'}(A).$$

This proves (2.22) (ii). $\square$

LEMMA 2.7. *Let Q be a q-open set. Then, for every $\xi \in Q$, there exists a q-open set Q_ξ such that*

$$\xi \in Q_\xi \subset \tilde{Q}_\xi \subset Q.$$

PROOF. By definition, every point in Q is a q-thin point of $E_0 = \partial\Omega \setminus Q$. Assume that $\operatorname{diam} Q < 1$ and put:

$$r_n = 2^{-n}, \quad K_n = \{\sigma : r_{n+1} \le |\sigma - \xi| \le r_n\}, \quad E_n := E_0 \cap K_n \cap \bar{B}_1(\xi).$$

Thus E_n is a q-closed set; we denote $E = \cup_{n=0}^{\infty} E_n$. Since ξ is a q-thin point of E,

$$\sum_0^\infty (r_n^{-N+1+2/(q-1)} C_{2/q,q'}(B_n \cap E))^{q-1} < \infty, \quad B_n = B_{r_n}(\xi),$$

which is equivalent to

$$\sum_0^\infty \left(r_n^{-N+1+2/(q-1)} C_{2/q,q'}(E_n)\right)^{q-1} < \infty.$$

Let $\{E_{m,n}\}_{m=1}^\infty$ be a q-proper stratification of E_n. Let $\bar{\epsilon}^n := \{\epsilon_{m,n}\}_{m=1}^\infty$ be a q-proper sequence (relative to the above stratification) such that $\epsilon_{1,n} \in (0, r_{n+2})$ and

$$C_{2/q,q'}(V_n) < 2C_{2/q,q'}(\cup E_n) \text{ where } V_n := \cup_{m=1}^\infty E_{m,n}^{\epsilon_{m,n}} \cap B_1(\xi).$$

Then $V_n \subset K_{n-2} \setminus K_{n+2}$, ξ is a q-thin point of the set $G = \cup_0^\infty V_n$ and $\xi \notin G$. Consequently $\xi \notin \tilde{G}$.

Put

$$Z_n := \cup_{m=1}^\infty \overline{E_{m,n}^{\epsilon_{m,n}/2}} \cap \bar{B}_{1/2}(\xi), \quad F_0 := \cup_0^\infty Z_n.$$

Since $Z_n \subset V_n$ it follows that ξ is a q-thin point of F_0 and $\xi \notin \tilde{F}_0$. Consequently $Q_0 := (Q \cap B_{1/2}(\xi)) \setminus \tilde{F}_0$ is a q-open subset of Q such that

$$\xi \in Q_0, \quad \tilde{Q}_0 \subset (\tilde{Q} \cap \bar{B}_{1/2}(\xi)) \setminus F_0 \subset (\tilde{Q} \cap \bar{B}_{1/2}(\xi)) \setminus E \subset Q.$$

□

3. Maximal solutions

We consider positive solutions of the equation (1.1) with $q \ge q_c$, in a bounded domain $\Omega \subset \mathbb{R}^N$ of class C^2. A function $u \in L^q_{loc}(\Omega)$ is a subsolution (resp. supersolution) of the equation if $-\Delta u + |u|^{q-1}u \le 0$ (resp. ≥ 0) in the distribution sense.

If $u \in L^q_{loc}(\Omega)$ is a subsolution of the equation then (by Kato's inequality [**8**]) $\Delta|u| \ge |u|^q$. Thus $|u|$ is subharmonic and consequently $u \in L^\infty_{loc}(\Omega)$. If $u \in L^q_{loc}(\Omega)$ is a solution then $u \in C^2(\Omega)$.

An increasing sequence of bounded domains of class C^2, $\{\Omega_n\}$, such that $\Omega_n \uparrow \Omega$ and $\bar{\Omega}_n \subset \Omega_{n+1}$ is called an *exhaustive* sequence relative to Ω.

PROPOSITION 3.1. *Let u be a non-negative function in $L^\infty_{loc}(\Omega)$.*

(i) If u is a subsolution of (1.1)*, there exists a minimal solution v dominating u, i.e., $u \le v \le U$ for any solution $U \ge u$.*

(ii) If u is a supersolution of (1.1)*, there exists a maximal solution w dominated by u, i.e., $V < w < u$ for any solution $V \le u$.*

All the inequalities above are a.e..

Proof. Let $u_\epsilon = J_\epsilon u$ where J_ϵ is a smoothing operator and u is extended by zero outside Ω. Put $\tilde{u} = \lim_{\epsilon\to 0} u_\epsilon$ (the limit exists a.e. in Ω and $\tilde{u} = u$ a.e.). Let β_0, Ω_β, Σ_β etc. be as in Notation 1.1. Since $u_\epsilon \to \tilde{u}$ in $L^1(\Omega)$ it follows that

$$u_\epsilon\big|_{\Sigma_\beta} \to \tilde{u}\big|_{\Sigma_\beta} \ \text{ in } \ L^1(\Sigma_\beta)$$

for a.e.$\beta \in (0,\beta_0)$. Choose a sequence $\{\beta_n\}$ decreasing to zero such that the above convergence holds for each surface $\Sigma_n := \Sigma_{\beta_n}$. Put $D_n := \Omega'_{\beta_n}$. Assuming that u is a subsolution of (1.1) in Ω, u_ϵ is a subsolution of the boundary value problem for (1.1) in D_n with boundary data $u_\epsilon\big|_{\Sigma_n}$. Consequently $\tilde{u}$ is a subsolution of the boundary value problem for (1.1) in D_n with boundary data $\tilde{u}\big|_{\Sigma_n} \in L^1(\Sigma_n)$. (Here we use the assumption $u \in L^\infty_{loc}(\Omega)$ in order to ensure that $u_\epsilon^q \to \tilde{u}^q$ in $L^1_{loc}(\Omega)$.) Let v_n denote the solution of this boundary value problem in the L^1 sense:

$$-\Delta v_n + v_n^q = 0 \ \text{ in } \ D_n, \qquad v_n = \tilde{u} \ \text{ on } \ \Sigma_n.$$

Then $v_n \in C^2(D_n) \cap L^\infty(D_n)$, $v_n \le \|u\|_{L^\infty(D_n)}$ and the boundary data is assumed in the L^1 sense. Clearly $\tilde{u} \le v_n$ in D_n, n=1,2,... . In particular, $v_n \le v_{n+1}$ on Σ_n. This implies $v_n \le v_{n+1}$ in D_n. In addition, by the Keller-Osserman inequality the sequence $\{v_n\}$ is eventually bounded in every compact subset of Ω. Therefore $v = \lim v_n$ is the solution with the properties stated in (i).

Next assume that u is a supersolution and let $\{D_n\}$ be as above. Since $u \in L^q(D_n)$ there exists a positive solution w_n of the boundary value problem

$$-\Delta w = u^q \text{ in } D_n, \quad w = 0 \text{ on } \Sigma_n.$$

Hence $u + w_n$ is superharmonic and its boundary trace is precisely $\tilde{u}\big|_{\Sigma_n}$. Consequently $u + w_n \ge z_n$ where z_n is the harmonic function in D_n with boundary data $\tilde{u}\big|_{\Sigma_n}$. Thus $u_n := z_n - w_n$ is the smallest solution of (1.1) in D_n dominating u. This implies that $\{u_n\}$ decreases and the limiting solution U is the smallest solution of (1.1) dominating u in Ω. □

Proposition 3.2. *Let u, v be non-negative, locally bounded functions in Ω.*

(i) If u, v are subsolutions (resp. supersolutions) then $\max(u,v)$ *is a subsolution (resp.* $\min(u,v)$ *is a supersolution).*

(ii) If u, v are supersolutions then $u + v$ is a supersolution.

(iii) If u is a subsolution and v a supersolution then $(u-v)_+$ is a subsolution.

Proof. The first two statements are well known; they can be verified by an application of Kato's inequality. The third statement is verified in a similar way:

$$\Delta(u-v)_+ = \operatorname{sign}_+(u-v)\Delta(u-v) \ge (u^q - v^q)_+ \ge (u-v)_+^q.$$

□

Notation 3.1. Let u, v be non-negative, locally bounded functions in Ω.

(a) If u is a subsolution, $[u]_\dagger$ denotes the smallest solution dominating u.

(b) If u is a supersolution, $[u]^\dagger$ denotes the largest solution dominated by u.

(c) If u, v are subsolutions then $u \vee v := [\max(u,v)]_\dagger$.

(d) If u, v are supersolutions then $u \wedge v := [\inf(u,v)]^\dagger$ and $u \oplus v := [u+v]^\dagger$.

(e) If u is a subsolution and v a supersolution then $u \ominus v := [(u-v)_+]_\dagger$.

The following result was proved in [**9**] (see also [**3**, Sec. 8.5]).

PROPOSITION 3.3. *(i) Let $\{u_k\}$ be a sequence of positive, continuous subsolutions of* (1.1)*. Then $U := \sup u_k$ is a subsolution. The statement remains valid if subsolution is replaced by supersolution and* sup *by* inf*.*

(ii) Let $\mathcal{T}$ be a family of positive solutions of (1.1)*. Suppose that, for every pair $u_1, u_2 \in \mathcal{T}$, there exists $v \in \mathcal{T}$ such that*

$$\max(u_1, u_2) \leq v, \quad \textit{resp.} \quad \min(u_1, u_2) \geq v.$$

Then there exists a monotone sequence $\{u_n\}$ in $\mathcal{T}$ such that

$$u_n \uparrow \sup \mathcal{T}, \quad \textit{resp.} \quad u_n \downarrow \sup \mathcal{T}.$$

Thus $\sup \mathcal{T}$ *(resp.* $\inf \mathcal{T}$*) is a solution.*

DEFINITION 3.4. A solution u of (1.1) vanishes on a relatively open set $Q \subset \partial\Omega$ if $u \in C(\Omega \cup Q)$ and $u = 0$ on Q. A positive solution u vanishes on a q-open set $A \subset \partial\Omega$ if

$$u = \sup\{v \in \mathcal{U}(\Omega) : \, v \leq u, \; v = 0 \text{ on some relatively open neighborhood of } A\}.$$

When this is the case we write $u \underset{A}{\approx} 0$.

LEMMA 3.5. *Let A be a q-open subset of $\partial\Omega$ and $u_1, u_2 \in \mathcal{U}(\Omega)$.*

(a) If both solutions vanish on A then $u_1 \vee u_2 \underset{A}{\approx} 0$. If $u_2 \underset{A}{\approx} 0$ and $u_1 \leq u_2$ then $u_1 \underset{A}{\approx} 0$.

(b) If $u \in \mathcal{U}(\Omega)$ and $u \underset{A}{\approx} 0$ then there exists an increasing sequence of solutions $\{u_n\} \subset \mathcal{U}(\Omega)$, each of which vanishes on a relatively open neighborhood of A (which may depend on n) such that $u_n \uparrow u$.

(c) If A, A' are q-open sets, $A \overset{q}{\sim} A'$ and $u \underset{A}{\approx} 0$ then $u \underset{A'}{\approx} 0$.

PROOF. The first assertion follows easily from the definition. Thus the set of solutions $\{v\}$ described in the definition is closed with respect to the binary operator $\vee$. Therefore, by Proposition 3.3, the supremum of this set is the limit of an increasing sequence of elements of this set.

The last statement is obvious. $\square$

DEFINITION 3.6. (a) Let $u \in \mathcal{U}(\Omega)$ and let A denote the union of all q-open sets on which u vanishes. Then $\partial\Omega \setminus A$ is called the *fine boundary support* of u, to be denoted by $\operatorname{supp}^q_{\partial\Omega} u$.

(b) For any Borel set E we denote

$$U_E = \sup\{u \in \mathcal{U}(\Omega) : u \underset{\tilde{E}^c}{\approx} 0, \; \tilde{E}^c = \partial\Omega \setminus \tilde{E}\}.$$

Thus $U_E = U_{\tilde{E}}$.

LEMMA 3.7. *(i) Let A be a q-open subset of $\partial\Omega$ and $\{u_n\} \subset \mathcal{U}(\Omega)$ a sequence of solutions vanishing on A. If $\{u_n\}$ converges then $u = \lim u_n$ vanishes on A. In particular, if E is Borel, U_E vanishes outside $\tilde{E}$.*

(ii) Let E be a Borel set such that $C_{2/q,q'}(E) = 0$. If $u \in \mathcal{U}(\Omega)$ and u vanishes on every q-open subset of $E^c = \partial\Omega \setminus E$ then $u \equiv 0$. In particular, $U_E \equiv 0$.

(iii) If $\{A_n\}$ is a sequence of Borel subsets of $\partial\Omega$ such that $C_{2/q,q'}(A_n) \to 0$ then $U_{A_n} \to 0$.

PROOF. (i) Using Lemma 3.5 we find that, in proving the first assertion, we may assume that $\{u_n\}$ is increasing. Now we can produce an increasing sequence of solutions $\{w_n\}$ such that, for each n, w_n vanishes on some (open) neighborhood of A and $\lim w_n = \lim u_n$. By definition $\lim w_n$ vanishes on A.

Let E be a q-closed set. By Lemma 3.5(a) and Proposition 3.3, there exists an increasing sequence of solutions $\{u_n\}$ vanishing outside E such that $U_E = \lim u_n$. Therefore U_E vanishes outside $\tilde{E}$.

(ii) Let A_n be open sets such that $E \subset A_n$, $A_n \downarrow$ and $C_{2/q,q'}(A_n) \to 0$. The sets $\tilde{A}_n$ have the same properties and, by assumption, u vanishes in $(\tilde{A}_n)^c := \partial\Omega \setminus \tilde{A}_n$. Therefore, for each n, there exists a solution w_n which vanishes on an open neighborhood B_n of $(\tilde{A}_n)^c$ such that $w_n \leq u$ and $w_n \to u$. Hence $w_n \leq U_{K_n}$ where $K_n = B_n^c$ is compact and $K_n \subset \tilde{A}_n$. Since $C_{2/q,q'}(K_n) \to 0$, the capacitary estimates of [**16**] imply that $\lim U_{K_n} = 0$ and hence $u = 0$.

(iii) By definition $U_{A_n} = U_{\tilde{A}_n}$. Therefore, in view of Proposition 2.2(iv), it is enough to prove the assertion when each set A_n is q-closed. As before, for each n, there exists a solution w_n which vanishes on an open neighborhood B_n of $(\tilde{A}_n)^c$ such that $w_n \leq U_{A_n}$ and $U_{A_n} - w_n \to 0$. Thus $w_n \leq U_{K_n}$ where $K_n = B_n^c$ is compact and $K_n \subset \tilde{A}_n$. Since $C_{2/q,q'}(K_n) \to 0$ it follows that $U_{K_n} \to 0$, which implies the assertion. $\square$

LEMMA 3.8. *Let E, F be Borel subsets of $\partial\Omega$.*

(i) If E, F are q-closed then $U_E \wedge U_F = U_{E\cap F}$.

(ii) If E, F are q-closed then

$$\begin{aligned} &U_E < U_F \iff [\,E \overset{q}{\subset} F \text{ and } C_{2/q,q'}(F \setminus E) > 0\,], \\ &U_E = U_F \iff E \overset{q}{\sim} F. \end{aligned} \tag{3.1}$$

(iii) If $\{F_n\}$ is a decreasing sequence of q-closed sets then

$$\lim U_{F_n} = U_F \quad \text{where} \quad F = \cap F_n. \tag{3.2}$$

(iv) Let $A \subset \partial\Omega$ be a q-open set and let $u \in \mathcal{U}(\Omega)$. Suppose that u vanishes q-locally in A, i.e., for every point $\sigma \in A$ there exists a q-open set A_σ such that

$$\sigma \in A_\sigma \subset A, \quad u \underset{A_\sigma}{\approx} 0.$$

Then u vanishes on A. In particular each solution $u \in \mathcal{U}(\Omega)$ vanishes on $\partial\Omega \setminus \operatorname{supp}^q_{\partial\Omega} u$.

PROOF. (i) $U_E \wedge U_F$ is the largest solution under $\inf(U_E, U_F)$ and therefore, by Definition 3.6, it is the largest solution which vanishes outside $E \cap F$.

(ii) Obviously

$$E \overset{q}{\sim} F \Longrightarrow U_E = U_F, \qquad E \overset{q}{\subset} F \iff U_E \leq U_F. \tag{3.3}$$

In addition,

$$C_{2/q,q'}(F \setminus E) > 0 \Longrightarrow U_E \neq U_F. \tag{3.4}$$

Indeed, if K is a compact subset of $F \setminus E$ of positive capacity, then $U_K > 0$ and $U_K \leq U_F$ but $U_K \not\leq U_E$. Therefore $U_F = U_E$ implies $F \overset{q}{\sim} E$.

(iii) If $V := \lim U_{F_n}$ then $U_F \leq V$. If $U_F < V$ then $C_{2/q,q'}(\operatorname{supp}^q_{\partial\Omega} V \setminus F) > 0$. But

$\operatorname{supp}^q_{\partial\Omega} V \subset F_n$ so that $\operatorname{supp}^q_{\partial\Omega} V \subset F$ and consequently $V \leq U_F$.

(iv) First assume that A is a countable union of q$-$*open* sets $\{A_n\}$ such that $u \underset{A_n}{\approx} 0$ for each n. Then u vanishes on $\cup_1^k A_i$ for each i. Therefore we may assume that the sequence $\{A_n\}$ is increasing. Put $F_n = \partial\Omega \setminus A_n$. Then $u \leq U_{F_n}$ and, by (iii), $U_{F_n} \downarrow U_F$ where $F = \partial\Omega \setminus A$. Thus $u \leq U_F$, i.e., which is equivalent to $u \underset{A}{\approx} 0$.

We turn to the general case. It is known that the (α, p)-fine topology possesses the quasi-Lindelöf property (see [**1**, Sec. 6.5.11]). Therefore A is covered, up to a set of capacity zero, by a countable subcover of $\{A_\sigma : \sigma \in A\}$. Therefore the previous argument implies that $u \underset{A}{\approx} 0$ □

THEOREM 3.9. *(a) Let E be a q-closed set. Then,*

$$\begin{aligned} U_E &= \inf\{U_D : E \subset D \subset \partial\Omega,\ D \text{ open}\} \\ &= \sup\{U_K : K \subset E,\ K \text{ compact}\}. \end{aligned} \tag{3.5}$$

(b) If E, F are two Borel subsets of $\partial\Omega$ then

$$U_E = U_{F\cap E} \oplus U_{E\setminus F}. \tag{3.6}$$

(c) Let E, F_n, $n = 1, 2, \dots$ be Borel subsets of $\partial\Omega$ and let u be a positive solution of (1.1)*. If either $C_{2/q,q'}(E\Delta F_n) \to 0$ or $F_n \downarrow E$ then*

$$U_{F_n} \to U_E. \tag{3.7}$$

PROOF. (a) Let $\{Q_j\}$ be a sequence of open sets, decreasing to a set $E' \overset{q}{\sim} E$, which satisfies (2.20). Then $\tilde{Q}_j \downarrow E'$ and, by Lemma 3.8(iii) $U_{Q_j} \downarrow U_E$. This implies the first equality in (3.5). The second equality follows directly from Definition 3.4 (see also Lemma 3.5).

(b) Let D, D' be open sets such that $\widetilde{E \cap F} \subset D$ and $\widetilde{E \setminus F} \subset D'$ and let K be a compact subset of $\tilde{E}$. Then

$$U_K \leq U_D + U_{D'}. \tag{3.8}$$

To verify this inequality, let v be a positive solution such that $\operatorname{supp}^q_{\partial\Omega} v \subset K$ and let $\{\beta_n\}$ be a sequence decreasing to zero such that the following limits exist:

$$w = \lim_{n\to\infty} v^D_{\beta_n}, \qquad w' = \lim_{n\to\infty} v^{D'}_{\beta_n}.$$

(See Notation 1.1 for the definition of v^D_β.) Then

$$v \leq w + w' \leq U_D + U_{D'}.$$

Since, by [**16**] $U_K = V_K$, this inequality implies (3.8). Further (3.8) and (3.5) imply

$$U_E \leq U_{F\cap E} + U_{E\setminus F}.$$

On the other hand, both $U_{F\cap E}$ and $U_{E\setminus F}$ vanish outside $\tilde{E}$. Consequently $U_{F\cap E} \oplus U_{E\setminus F}$ vanishes outside $\tilde{E}$ so that

$$U_E \geq U_{F\cap E} \oplus U_{E\setminus F}.$$

This implies (3.6).

(c) The previous statement implies,

$$U_E \leq U_{F_n\cap E} + U_{E\setminus F_n}, \qquad U_{F_n} \leq U_{F_n\cap E} + U_{F_n\setminus E}.$$

If $C_{2/q,q'}(E\Delta F_n) \to 0$, Lemma 3.7 implies $\lim U_{E\Delta F_n} = 0$ which in turn implies (3.7).

If $F_n \downarrow E$ then, by Lemma 3.8, $U_{F_n} \to U_E$. $\square$

Notation 3.2. For any Borel set $E \subset \partial\Omega$ of positive $C_{2/q,q'}$-capacity put

$$\begin{aligned} &\mathcal{V}_{mod}(E) = \{u_\mu : \mu \in W_+^{-2/q,q}(\partial\Omega),\ \mu(\partial\Omega \setminus E) = 0\}, \\ &V_E = \sup\, \mathcal{V}_{mod}(E) \end{aligned} \tag{3.9}$$

THEOREM 3.10. *If E is a q-closed set, then*

$$U_E = V_E. \tag{3.10}$$

Thus the maximal solution U_E is σ-moderate. Furthermore U_E satisfies the capacitary estimates established in [**16**] *for compact sets, namely:*

There exist positive constants c_1, c_2 depending only on q, N and Ω such that, for every $x \in \Omega$,

$$\begin{aligned} &c_2\rho(x) \sum_{m=-\infty}^{\infty} r_m^{-1-2/(q-1)} C_{2/q,q'}((E \cap S_m(x))/r_m)) \le U_E(x) \le \\ &c_1\rho(x) \sum_{m=-\infty}^{\infty} r_m^{-1-2/(q-1)} C_{2/q,q'}((E \cap S_m(x))/r_m)), \end{aligned} \tag{3.11}$$

where

$$\rho(x) = \mathrm{dist}\,(x, \partial\Omega), \quad r_m := 2^{-m},\ S_m(x) = \{y \in \partial\Omega :\ r_{m+1} \le |x-y| \le r_m\}.$$

Note that, for each point x, $S_m(x) = \emptyset$ when

$$\sup_{y\in E} |x-y| < r_{m+1} < r_m < \rho(x).$$

Therefore the sum is finite for each $x \in \Omega$.

Remark. Actually the estimates hold for any Borel set E. Indeed, by definition, $U_E = U_{\tilde{E}}$ and $C_{2/q,q'}((E \cap S_m(x))/r_m)) \sim C_{2/q,q'}((\tilde{E} \cap S_m(x))/r_m))$.

PROOF. Let $\{E_k\}$ be a q-stratification of E. If $u \in \mathcal{V}_{mod}(E)$ and $\mu = \mathrm{tr}\, u$ then $u_\mu = \sup u_{\mu_k}$ where $\mu_k = \mu\chi_{E_k}$. Hence $V_E = \sup V_{E_k}$. By [**16**], $U_{E_k} = V_{E_k}$. These facts and Theorem 3.9(c) imply (3.10). It is known that U_{E_k} satisfies the capacitary estimates (3.11). In addition,

$$C_{2/q,q'}((E_k \cap S_m(x))/r_m)) \to C_{2/q,q'}((E \cap S_m(x))/r_m)).$$

Therefore U_E satisfies the capacitary estimates. $\square$

4. Localization

DEFINITION 4.1. Let μ be a positive bounded Borel measure on $\partial\Omega$ which vanishes on sets of $C_{2/q,q'}$-capacity zero.

(a) The q-support of μ (denoted q-supp μ) is the intersection of all q-closed sets F such that $\mu(\partial\Omega \setminus F) = 0$.

(b) We say that μ is concentrated on a Borel set E if $\mu(\partial\Omega \setminus E) = 0$.

LEMMA 4.2. *If μ is a measure as in the previous definition then,*

$$q\text{-}supp\,\mu \overset{q}{\sim} \mathrm{supp}^q_{\partial\Omega}\, u_\mu. \tag{4.1}$$

PROOF. Put $F = \mathrm{supp}^q_{\partial\Omega}\, u_\mu$. By Lemma 3.8(iv) u_μ vanishes on $\partial\Omega \setminus F$ and by Lemma 3.5 there exists an increasing sequence of positive solutions $\{u_n\}$ such that each function u_n vanishes outside a compact subset of F, say F_n, and $u_n \uparrow u_\mu$. If $S_n := \mathrm{supp}^q_{\partial\Omega}\, u_n$ then $S_n \subset F_n$ and $\{S_n\}$ increases. Thus $\{\bar{S}_n\}$ is an increasing sequence of compact subsets of F and, setting $\mu_n = \mu\chi_{\bar{S}_n}$, we find $u_n \leq u_{\mu_n} \leq u_\mu$ so that $u_{\mu_n} \uparrow u_\mu$. This, in turn, implies (see [**14**])

$$\mu_n \uparrow \mu, \quad q\text{-supp}\,\mu \overset{q}{\subset} \widetilde{\cup_n \bar{S}_n} \subset F.$$

If D is a relatively open set and $\mu(D) = 0$ it is clear that u_μ vanishes on D. Therefore u_{μ_n} vanishes outside $\bar{S}_n$, thus outside q-supp μ. Consequently u_μ vanishes outside q-supp μ, i.e. $F \overset{q}{\subset} q\text{-supp}\,\mu$. $\square$

DEFINITION 4.3. Let u be a positive solution and A a Borel subset of $\partial\Omega$. Put

$$[u]_A := \sup\{v \in \mathcal{U}(\Omega) : \; v \leq u, \quad \mathrm{supp}^q_{\partial\Omega}\, v \overset{q}{\subset} \tilde{A}\} \tag{4.2}$$

and,

$$[u]^A := \sup\{[u]_F : F \overset{q}{\subset} A, \; F \; q\text{-closed}\}. \tag{4.3}$$

Thus $[u]_A = u \wedge U_A$, i.e., $[u]_A$ is the largest solution under $\inf(u, U_A)$.

Recall that, if A is q-open and $u \in C(\partial\Omega)$, u^A_β denotes the solution of (1.1) in Ω'_β which equals $u\chi_{\Sigma_\beta(A)}$ on Σ_β.

If $\lim_{\beta\to 0} u^A_\beta$ exists the limit will be denoted by u^A.

THEOREM 4.4. *Let $u \in \mathcal{U}(\Omega)$.*
(i) If $E \subset \partial\Omega$ is q-closed then,

$$[u]_E = \inf\{[u]_D : E \subset D \subset \partial\Omega, \; D \textit{ open}\}. \tag{4.4}$$

(ii) If E, F are two Borel subsets of $\partial\Omega$ then

$$[u]_E \leq [u]_{F\cap E} + [u]_{E\setminus F} \tag{4.5}$$

and

$$[[u]_E]_F = [[u]_F]_E = [u]_{E\cap F}. \tag{4.6}$$

(iii) Let E, F_n, $n = 1, 2, \ldots$ be Borel subsets of $\partial\Omega$. If either $C_{2/q,q'}(E\Delta F_n) \to 0$ or $F_n \downarrow E$ then

$$[u]_{F_n} \to [u]_E. \tag{4.7}$$

PROOF. (i) Let $\mathcal{D} = \{D\}$ be the family of sets in (4.4). By (3.5) (with respect to the family $\mathcal{D}$)

$$\inf(u, U_E) = \inf(u, \inf_{D\in\mathcal{D}} U_D) = \inf_{D\in\mathcal{D}} \inf(u, U_D) \geq \inf_{D\in\mathcal{D}} [u]_D. \tag{4.8}$$

Obviously

$$[u]_{D_1} \wedge [u]_{D_2} \geq [u]_{D_1\cap D_2}.$$

(In fact we have equality but that is not needed here.) Therefore, by Proposition 3.3, the function $v := \inf_{D\in\mathcal{D}} [u]_D$ is a solution of (1.1). Hence (4.8) implies $[u]_E \geq v$. The opposite inequality is obvious.

(ii) If E is compact (4.5) is proved in the same way as Theorem 3.9(b). In general, if $\{E_n\}$ is a q stratification of $\tilde{E}$,

$$[u]_{E_n} \leq [u]_{F\cap E_n} + [u]_{E_n\setminus F} \leq [u]_{F\cap E} + [u]_{E\setminus F}.$$

This inequality and Theorem 3.9(c) imply (4.5).

Put $A = \tilde{E}$ and $B = \tilde{F}$. It follows directly from the definition that,

$$[[u]_A]_B \leq inf(u, U_A, U_B).$$

The largest solution dominated by u and vanishing on $A^c \cup B^c$) is $[u]_{A\cap B}$. Thus

$$[[u]_A]_B \leq [u]_{A\cap B}.$$

On the other hand

$$[u]_{A\cap B} = [[u]_{A\cap B}]_B \leq [[u]_A]_B.$$

This proves (4.6). (iii) By (4.8)

$$[u]_E \leq [u]_{F_n\cap E} + [u]_{E\setminus F_n}, \qquad [u]_{F_n} \leq [u]_{F_n\cap E} + [u]_{F_n\setminus E}.$$

If $C_{2/q,q'}(E\Delta F_n) \to 0$, Lemma 3.7 implies $\lim[u]_{E\Delta F_n} = 0$ which in turn implies (4.7).

If $F_n \downarrow E$ then, by Lemma 3.8, $U_{F_n} \to U_E$. If u is a positive solution then

$$\inf(u, U_E) = \inf(u, \inf_n U_{F_n}) = \inf_n \inf(u, U_{F_n}) \geq \inf_n [u]_{F_n}.$$

Since $\{F_n\}$ decreases $w = \inf_n [u]_{F_n}$ is a solution. Hence $[u]_E \geq w$. The opposite inequality is obvious; hence $[u]_E = \lim[u]_{F_n}$. □

LEMMA 4.5. *Let u be a positive solution of* (1.1) *and put $E = \operatorname{supp}^q_{\partial\Omega} u$.*

(i) If D is a q-open set such that $E \overset{q}{\subset} D$ then

$$[u]^D = \lim_{\beta\to 0} u^D_\beta = [u]_D = u. \tag{4.9}$$

(ii) If A is a q-open subset of $\partial\Omega$,

$$u \underset{A}{\approx} 0 \iff u^Q = \lim_{\beta\to 0} u^Q_\beta = 0 \quad \forall Q \ q\text{-open} : \tilde{Q} \overset{q}{\subset} A. \tag{4.10}$$

(iii) Finally,

$$u \underset{A}{\approx} 0 \iff [u]^A = 0 \tag{4.11}$$

PROOF. *Case 1: E is closed.* Since u vanishes in $A := \partial\Omega \setminus E$, it follows that $u \in C(\Omega \cup A)$ and $u = 0$ on A. If, in addition, $D \subset \partial\Omega$ is an *open* neighborhood of E then

$$\int_{\Sigma_\beta(D^c)} udS \to 0$$

so that

$$\lim_{\beta\to 0} u^{D^c}_\beta = 0. \tag{4.12}$$

Since

$$u^D_\beta \leq u \leq u^D_\beta + u^{D^c}_\beta \quad \text{in} \ \ \Omega'_\beta$$

it follows that

$$u = \lim u^D_\beta. \tag{4.13}$$

If we assume only that D is q-open and $E \overset{q}{\subset} D$ then, for every $\epsilon > 0$, there exists an open set O_ϵ such that $D \subset O_\epsilon$, $E \subset O_\epsilon$ and $C_{2/q,q'}(O'_\epsilon) < \epsilon$ where $O'_\epsilon = O_\epsilon \setminus D$. It follows that

$$u_\beta^{O_\epsilon} - u_\beta^D \le U_{\Sigma_\beta(O'_\epsilon)} \text{ in } \Omega'_\beta$$

and $\lim_{\epsilon\to 0} U_{\Sigma_\beta(O'_\epsilon)} = 0$ uniformly with respect to β. Since $\lim_{\beta\to 0} u_\beta^{O_\epsilon} = u$ it follows that (4.13) holds. The same argument shows that (4.12) remains valid.

Now (4.13) implies (4.9). Indeed

$$u = \lim u_\beta^D \le [u]_D \le u.$$

Hence $u = [u]_D$. If Q is a q-open set such that $E \overset{q}{\subset} Q \subset \tilde{Q} \overset{q}{\subset} D$ then $u = [u]_Q \le [u]^D$. Hence $u = [u]^D$.

In addition (4.12) implies (4.10) in the direction $\Longrightarrow$. Assertion (4.10) in the opposite direction is a consequence of Lemma 2.7 and Lemma 3.8 (iv).

Case 2. We consider the general case when E is q-closed. Let $\{E_n\}$ be a stratification of E so that $C_{2/q,q'}(E \setminus E_n) \to 0$. If D is q-open and $E \overset{q}{\subset} D$ then, by the first part of the proof,

$$\lim_{\beta\to 0}([u]_{E_n})_\beta^D = [u]_{E_n}. \tag{4.14}$$

By (4.5)

$$u_\beta^D \le ([u]_{E_n})_\beta^D + ([u]_{E\setminus E_n})_\beta^D. \tag{4.15}$$

Let $\{\beta_k\}$ be a sequence decreasing to zero such that the following limits exist

$$w := \lim_{k\to 0} u_{\beta_k}^D, \quad w_n := \lim_{k\to 0}([u]_{E\setminus E_n})_{\beta_k}^D, \quad n = 1, 2, \cdots.$$

Then, by (4.14) and (4.15),

$$[u]_{E_n} \le w \le [u]_{E_n} + w_n \le [u]_{E_n} + U_{E\setminus E_n}.$$

Further, by (3.7),

$$[u]_{E_n} \to [u]_E = u, \quad U_{E\setminus E_n} \to 0.$$

Hence $w = u$. This implies (4.13) which in turn implies (4.9).

To verify (4.10) in the direction $\Longrightarrow$ we apply (4.15) with D replaced by Q. We obtain,

$$u_\beta^Q \le ([u]_{E_n})_\beta^Q + ([u]_{E\setminus E_n})_\beta^Q.$$

By the first part of the proof

$$\lim_{\beta\to 0}([u]_{E_n})_\beta^Q = 0.$$

Let $\{\beta_k\}$ be a sequence decreasing to zero such that the following limits exist:

$$\lim_{k\to\infty}([u]_{E\setminus E_n})_{\beta_k}^Q, \quad n = 1, 2, \ldots, \quad \lim_{k\to\infty} u_{\beta_k}^Q.$$

Then

$$\lim_{k\to\infty} u_{\beta_k}^Q \le \lim_{k\to\infty}([u]_{E\setminus E_n})_{\beta_k}^Q \le U_{E\setminus E_n}.$$

Since $U_{E\setminus E_n} \to 0$ we obtain (4.10) in the direction $\Longrightarrow$. The assertion in the opposite direction is proved as in Case 1. This completes the proof of (i) and (ii).

Finally we prove (iii). First assume that $u \underset{A}{\approx} 0$. If F is a q-closed set such that $F \overset{q}{\subset} A$ then there exists a q-open set Q such that $F \overset{q}{\subset} \tilde{Q} \overset{q}{\subset} A$. Therefore, applying (4.9) to $v := [u]_F$ and using (4.10) we obtain

$$v = \lim v_\beta^Q \leq \lim u_\beta^Q = 0.$$

In view of Definition 4.3 this implies that $[u]^A = 0$.

Secondly assume that $[u]^A = 0$. Then $[u]_Q = 0$ whenever $\tilde{Q} \overset{q}{\subset} A$. If Q is a q-open set such that $\tilde{Q} \overset{q}{\subset} A$ then $[u]_Q = 0$ and hence $u \underset{Q}{\approx} 0$. Applying once again Lemma 2.7 and Lemma 3.8 (iv) we conclude that $u \underset{A}{\approx} 0$. □

Definition 4.6. Let u, v be positive solutions of (1.1) in Ω and let A be a q-open subset of $\partial\Omega$. We say that $u = v$ on A if $u \ominus v$ and $v \ominus u$ vanish on A (see Notation 3.1). This relation is denoted by $u \underset{A}{\approx} v$.

Theorem 4.7. *Let $u, v \in \mathcal{U}(\Omega)$ and let A be a q-open subset of $\partial\Omega$. Then,*

$$u \underset{A}{\approx} v \Longleftrightarrow \lim_{\beta\to 0} |u - v|_\beta^Q = 0, \tag{4.16}$$

for every q-open set Q such that $\tilde{Q} \overset{q}{\subset} A$ and

$$u \underset{A}{\approx} v \iff [u]_F = [v]_F, \tag{4.17}$$

for every q-closed set F such that $F \overset{q}{\subset} A$.

Proof. By definition, $u \underset{A}{\approx} v$ is equivalent to $u \ominus v \underset{A}{\approx} 0$ and $v \ominus u \underset{A}{\approx} 0$. Hence, by Lemma 4.5 (specifically (4.11)),

$$[u \ominus v]_F = 0, \quad [v \ominus u]_F = 0, \tag{4.18}$$

for every q-closed set $F \overset{q}{\subset} A$. Therefore, if $\tilde{Q} \overset{q}{\subset} A$, Lemma 4.5 implies that

$$((u - v)_+)_\beta^Q \to 0, \quad ((v - u)_+)_\beta^Q \to 0.$$

(Recall that $u \ominus v$ is the smallest solution which dominates the subsolution $(u-v)_+$.) This implies (4.16) in the direction $\Longrightarrow$; the opposite direction is a consequence of Lemma 3.7.

We turn to the proof of (4.17). For any two positive solutions u, v we have

$$u + (v - u)_+ \leq v + (u - v)_+ \leq v + u \ominus v. \tag{4.19}$$

If F is a q-closed set and Q a q-open set such that $F \overset{q}{\subset} Q$ then,

$$[u]_F \leq [v]_Q + [u \ominus v]_Q. \tag{4.20}$$

To verify this inequality we observe that, by (4.19),

$$[u]_F \leq [v]_Q + [v]_{Q^c} + [u \ominus v]_Q + [u \ominus v]_{Q^c}.$$

The subsolution $w := ([u]_F - ([v]_Q + [u \ominus v]_Q))_+$ is dominated by the supersolution $[v]_{Q^c} + [u \ominus v]_{Q^c}$ which vanishes on Q. Therefore w vanishes on Q. Since the boundary support of $[w]_\dagger$ is contained in F it follows that $[w]_\dagger \equiv 0$ so that $w \equiv 0$.

If $u \underset{A}{\approx} v$ and $F \overset{q}{\subset} Q \subset \tilde{Q} \overset{q}{\subset} A$ then (4.20) and (4.18) imply,

$$[u]_F \leq [v]_Q.$$

Choosing a decreasing sequence of q-open sets $\{Q_n\}$ such that $\cap Q_n \overset{q}{\sim} F$ we obtain $[u]_F \leq \lim[v]_{Q_n} = [v]_F$. Similarly, $[v]_F \leq [u]_F$ and hence equality.

Next assume that $[v]_F = [u]_F$ for every q-closed set $F \overset{q}{\subset} A$. If Q is a q-open set such that $\tilde{Q} \overset{q}{\subset} A$ we have,

$$u \ominus v \leq ([u]_Q \oplus [u]_{Q^c}) \ominus [v]_Q \leq [u]_{Q^c},$$

because $[u]_Q = [v]_Q$. This implies that $u \ominus v$ vanishes on Q. Since this holds for every Q as above it follows that $u \ominus v$ vanishes on A. Similarly $v \ominus u$ vanishes on A. □

Corollary 4.8. *If A is a q-open subset of $\partial\Omega$, the relation $\underset{A}{\approx}$ is an equivalence relation in $\mathcal{U}(\Omega)$.*

Proof. This is an immediate consequence of (4.16). □

5. The precise boundary trace

5.1. The regular boundary set. We define the regular boundary set of a positive solution of (1.1) and present some conditions for the regularity of a q-open set.

Definition 5.1. Let u be a positive solution of (1.1).

a: Let $D \subset \partial\Omega$ be a q-open set such that $C_{2/q,q'}(D) > 0$. D is *pre-regular* with respect to u if

$$\int_\Omega [u]_F^q \rho dx < \infty \quad \forall F \overset{q}{\subset} D,\ F\ q\text{-closed}. \tag{5.1}$$

b: An arbitrary Borel set E is *regular* if there exists a pre-regular set D such that $\tilde{E} \overset{q}{\subset} D$.

c: A set $D \subset \partial\Omega$ is *σ-regular* if it is the union of a countable family of pre-regular sets.

d: The union of all q-open regular sets is called *the regular boundary set of* u, and is denoted by $\mathcal{R}(u)$. The set $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is called *the singular boundary set* of u. A point $P \in \mathcal{R}(u)$ is called a regular boundary point of u; a point $P \in \mathcal{S}(u)$ is called a singular boundary point of u.

Remark. The property of regularity of a set is preserved under the equivalence relation $\overset{q}{\sim}$. However note that *a point is regular if and only if it has a q-open regular neighborhood.*

Lemma 5.2. *If D is a q-open pre-regular set then every point $\xi \in D$ is a regular point. Furthermore there exists a q-open regular set Q such that*

$$\xi \in Q \subset \tilde{Q} \subset D. \tag{5.2}$$

If F is a regular q-closed set then there exists a regular q-open set Q such that $F \overset{q}{\subset} Q$.

Proof. By Lemma 2.7, for every $\xi \in D$, there exists a q-open set Q such that (5.2) holds. Therefore Q is a regular set and ξ is a regular point. The last assertion is a consequence of Lemma 2.4. □

DEFINITION 5.3. Let u be a positive solution of (1.1) and let $\{Q_n\}$ be an increasing sequence of regular q-open sets. If $\tilde{Q}_n \overset{q}{\subset} Q_{n+1}$ we say that $\{Q_n\}$ is *a regular sequence* relative to u.

If Q is a q-open set, $\{Q_n\}$ is a regular sequence relative to u, $Q_n \subset Q$ and $Q_0 := \cup_{n=1}^{\infty} Q_n \overset{q}{\sim} Q$ we refer to Q_0 as a *proper representation* of Q and to $\{Q_n\}$ as a *regular decomposition* of Q, relative to u.

LEMMA 5.4. *Let $u \in \mathcal{U}(\Omega)$. A q-open set $Q \subset \partial\Omega$ is σ-regular if and only if it has a proper representation relative to u. In particular every pre-regular set has a proper representation.*

PROOF. The 'if' direction follows immediately from the definition. Now suppose that Q is σ-regular. Then $Q = \cup_1^{\infty} E_n$ where E_n is q-open and pre-regular, $n = 1, 2, \cdots$. By Lemma 2.6, each set E_n can be represented (up to a set of capacity zero) as a countable union of q-open sets $\{A_{n,j}\}_{j=1}^{\infty}$ such that $\tilde{A}_{n,j} \overset{q}{\subset} A_{n,j+1} \overset{q}{\subset} E_n$. We may assume that $A_{n,j} \subset E_n$; otherwise we replace it by $A_{n,j} \cap E_n$. Put

$$Q_n = \cup_{k+j=n} A_{k,j}.$$

If $k + j = n$ then $\tilde{A}_{k,j} \overset{q}{\subset} A_{k,j+1} \subset Q_{n+1}$. Hence

$$\tilde{Q}_n \overset{q}{\subset} Q_{n+1}, \quad Q_0 := \cup Q_n \overset{q}{\sim} Q.$$

$\square$

THEOREM 5.5. *Let D be a q-open set such that $C_{2/q,q'}(D) > 0$.*

(i) *Suppose that*

$$\liminf_{\beta \to 0} \int_{\Omega'_\beta} (u_\beta^D)^q (\rho - \beta)\, dx < \infty. \tag{5.3}$$

Then D is pre-regular.

(ii) Suppose that D is a pre-regular set. Then there exists a Borel measure μ on D such that, for every q-closed set $E \overset{q}{\subset} D$,

$$\operatorname{tr}[u]_E = \mu\chi_{_E}. \tag{5.4}$$

PROOF. (i) Let $\{\beta_n\}$ be a sequence decreasing to zero such that

$$\int_{\Omega'_{\beta_n}} (u_{\beta_n}^D)^q (\rho - \beta_n)\, dx \leq C < \infty, \quad \forall n. \tag{5.5}$$

By extracting a subsequence if necessary we may assume that $\{u_{\beta_n}^D\}$ converges locally uniformly in Ω to a solution w. Then, by Lemma 4.5, if E is q-closed and $E \overset{q}{\subset} D$,

$$[u]_E = \lim_{\beta \to 0} ([u]_E)_\beta^D \leq \lim_{n \to \infty} u_{\beta_n}^D =: w. \tag{5.6}$$

By (5.5) and Fatou's lemma,

$$\int_\Omega w^q \rho\, dx \leq C < \infty.$$

Hence, by (5.6),

$$\int_\Omega ([u]_E)^q \rho\, dx \leq C < \infty. \tag{5.7}$$

Thus D is pre-regular.
(ii) By Lemma 5.4, D possesses a regular decomposition $\{D_j\}$. Put

$$w_j = [u]_{D_j}.$$

Then $\{w_j\}$ is increasing and its limit is a solution $w_0 \le w$ with w as defined in (5.6). Thus w_0 is a moderate solution. If $E \overset{q}{\subset} D$ is a q-closed set then, by (4.6),

$$[u]_{E\cap\tilde{D}_j} = [w_j]_E \le [w_0]_E \le [u]_E.$$

By Lemma 2.5, for every $k \in \mathbb{N}$ there exists an open set O_k and a natural number j_k such that $C_{2/q,q'}(O_k) < 1/k$ and $E \setminus O_k \overset{q}{\subset} D_{j_k}$. By Theorem 3.9

$$[u]_E \le [u]_{E\cap\tilde{D}_{j_k}} + [u]_{O_k}.$$

Since $[u]_{O_k} \to 0$ we conclude that

$$[w_0]_E = [u]_E \quad \forall E \overset{q}{\subset} D : \ E \ q\text{-closed}. \tag{5.8}$$

If w_0 is moderate then $\mathrm{tr}\,[w_0]_E = \mu\chi_{_E}\mathrm{tr}\,w_0$, which implies (5.4).

We turn to the case where w_0 is not moderate. The solution w_j is moderate and we denote

$$\mu_j = \mathrm{tr}\,w_j, \quad \mu = \lim \mu_j.$$

By (4.6), $w_j = [w_{j+k}]_{D_j}$. Therefore $\mu_j = \mu_{j+k}\chi_{_{D_j}} = \mu\chi_{_{D_j}}$. Therefore if E is q-closed and $E \overset{q}{\subset} D_j$ for some j, (5.4) holds with μ as defined above. If E is q-closed and $E \overset{q}{\subset} D$ then $E \overset{q}{\sim} E' := \cup(E \cap D_j) = \cup(E\cap\tilde{D}_j)$. Put $E_j := E\cap\tilde{D}_j$. It follows that

$$\mathrm{tr}\,[u]_{E_j} = \mu\chi_{_{E_j}} \uparrow \mu\chi_{_{E'}}.$$

Since D is pre-regular, $[u]_E$ is moderate. Put $E'_j = E \setminus D_j$ and observe that $\cap_1^\infty E'_j$ is a set of capacity zero so that (by Lemma 3.8) $U_{E'_j} \downarrow 0$ and hence $\lim[u]_{E'_j} \downarrow 0$. Since

$$[u]_E \le [u]_{E_j} + [u]_{E'_j} \ \text{ and } \ [u]_{E'_j} \downarrow 0$$

we conclude that

$$\mathrm{tr}\,[u]_E \le \lim \mathrm{tr}\,[u]_{E_j} = \mu\chi_{_{E'}}.$$

On the other hand

$$\mathrm{tr}\,[u]_E \ge \mathrm{tr}\,[u]_{E_j} \to \mu\chi_{_{E'}} = \mu\chi_{_E}.$$

This implies (5.4). □

COROLLARY 5.6. *Let D be a q-open set such that $C_{2/q,q'}(D) > 0$. Suppose that, for every q-open set Q such that $\tilde{Q} \overset{q}{\subset} D$,*

$$\liminf_{\beta\to 0} \int_{\Omega'_\beta} (u^Q_\beta)^q(\rho - \beta)\,dx < \infty. \tag{5.9}$$

Then D is pre-regular.

PROOF. This is an immediate consequence of Lemma 2.4 and Theorem 5.5. □

5.2. Behavior of the solution at the boundary. In this subsection we provide a characterization of regular and singular boundary points of a positive solution by the limiting behavior of the solution as it approaches the boundary in a q-open neighborhood of each point.

THEOREM 5.7. *Let u be a positive solution of* (1.1) *in* Ω *and let Q be a q-open subset of $\partial\Omega$ of positive capacity.*
(i) If $\xi \in \mathcal{S}(u)$ then, for every nontrivial q-open neighborhood Q of ξ,

$$\int_{\Sigma_\beta(Q)} udS \to \infty, \quad \textit{as} \quad \beta \to 0. \tag{5.10}$$

(ii) If $\xi \in \mathcal{R}(u)$, there exists a q-open regular set D such that $\xi \in D$. Further there exists a q-open set Q such that $\xi \in Q \subset \tilde{Q} \subset D$. Consequently

$$\sup_{0<\beta<\beta_0} \int_{\Sigma_\beta(Q)} udS < \infty, \qquad u^Q = \lim_{\beta\to 0} u^Q_\beta \ \textit{exists} \tag{5.11}$$

and u^Q is moderate. If $\mu_Q := \operatorname{tr}[u]_Q$ then, for every q-closed set $E \overset{q}{\subset} Q$,

$$\operatorname{tr}[u]_E = \mu_Q \chi_E. \tag{5.12}$$

PROOF. (i) If Q is a q-open set for which (5.10) does not hold, there exists a sequence $\{\beta_n\}$ converging to zero such that

$$\int_{\Sigma_{\beta_n}(Q)} udS \to \alpha < \infty.$$

This implies that there exists a constant C such that (5.5) holds (with D replaced by Q). By Theorem 5.5, Q is pre-regular and by Lemma 5.2 every point in Q is a regular point. Therefore, if $\xi \in \mathcal{S}(u)$, (5.10) holds.
(ii) By Definition 5.1 there exists a q-open regular neighborhood D of ξ. By Lemma 2.7 there exists a q-open set Q such that $\xi \in Q \subset \tilde{Q} \subset D$. By Theorem 4.4 and Lemma 4.5,

$$u \leq [u]_D + [u]_{D^c} \quad \text{and} \quad \lim_{\beta\to 0} \int_{\tilde{Q}} [u]_{D^c}(\beta, \cdot)dS = 0.$$

Therefore

$$\limsup_{\beta\to 0} \int_Q u(\beta, \cdot)dS \leq \limsup_{\beta\to 0} \int_Q [u]_D(\beta, \cdot)dS < \infty$$

so that (5.3) holds. In view of this fact, Theorem 5.5 and the arguments in its proof imply assertion (ii). □

5.3. q-perfect measures.

DEFINITION 5.8. Let μ be a positive Borel measure, not necessarily bounded, on $\partial\Omega$.
(i) We say that μ is *essentially absolutely continuous* relative to $C_{2/q,q'}$ if the following condition holds:
If Q is a q-open set and A is a Borel set such that $C_{2/q,q'}(A) = 0$ then

$$\mu(Q \setminus A) = \mu(Q).$$

This relation will be denoted by $\mu \underset{f}{\prec\prec} C_{2/q,q'}$.

(ii) μ is *regular relative to q-topology* if, for every Borel set $E \subset \partial\Omega$,

$$\begin{aligned}\mu(E) &= \inf\{\mu(D) : E \subset D \subset \partial\Omega,\ D\ q\text{-open}\} \\ &= \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.\end{aligned} \tag{5.13}$$

μ is *outer regular relative to q-topology* if the first equality in (5.13) holds.

(iii) A positive Borel measure is called *q-perfect* if it is essentially absolutely continuous relative to $C_{2/q,q'}$ and outer regular relative to q-topology. The space of q-perfect Borel measures is denoted by $\mathbb{M}_q(\partial\Omega)$.

Lemma 5.9. *If $\mu \in \mathbb{M}_q(\partial\Omega)$ and $A \subset \partial\Omega$ is a non-empty Borel set such that $C_{2/q,q'}(A) = 0$ then*

$$\mu(A) = \begin{cases} \infty & \text{if } \mu(Q \setminus A) = \infty \quad \forall Q\ q\text{-open neighborhood of } A, \\ 0 & \text{otherwise}. \end{cases} \tag{5.14}$$

If μ_0 is an essentially absolutely continuous positive Borel measure on $\partial\Omega$ and Q is a q-open set such that $\mu_0(Q) < \infty$ then $\mu_0\big|_Q$ is absolutely continuous with respect to $C_{2/q,q'}$ in the strong sense, i.e., if $\{A_n\}$ is a sequence of Borel subsets of $\partial\Omega$,

$$C_{2/q,q'}(A_n) \to 0 \Longrightarrow \mu_0(Q \cap A_n) \to 0.$$

Let μ_0 be an essentially absolutely continuous positive Borel measure on $\partial\Omega$. Put

$$\mu(E) := \inf\{\mu_0(D) : E \subset D \subset \partial\Omega,\ D\ q\text{-open}\}, \tag{5.15}$$

for every Borel set $E \subset \partial\Omega$. Then

$$\begin{aligned} &(a) \quad \mu_0 \le \mu, \qquad \mu_0(Q) = \mu(Q) \quad \forall Q \quad q\text{-open} \\ &(b) \quad \mu\big|_Q = \mu_0\big|_Q \ \text{for every } q\text{-open set } Q \text{ such that } \mu_0(Q) < \infty\ . \end{aligned} \tag{5.16}$$

Finally μ is q-perfect; thus μ is the smallest measure in $\mathbb{M}_q$ which dominates μ_0.

Proof. The first assertion follows immediately from the definition of $\mathbb{M}_q$. We turn to the second assertion. If μ_0 is an essentially absolutely continuous positive Borel measure on $\partial\Omega$ and Q is a q-open set such that $\mu_0(Q) < \infty$ then $\mu_0\chi_Q$ is a bounded Borel measure which vanishes on sets of $C_{2/q,q'}$-capacity zero. If $\{A_n\}$ is a sequence of Borel sets such that $C_{2/q,q'}(A_n) \to 0$ and $\mu_n := \mu_0\chi_{Q\cap A_n}$ then $u_{\mu_n} \to 0$ locally uniformly and $\mu_n \rightharpoonup 0$ weakly with respect to $C(\partial\Omega)$. Hence $\mu_0(Q \cap A_n) \to 0$. Thus μ_0 is absolutely continuous in the strong sense relative to $C_{2/q,q'}$.

Assertion (5.16) (a) follows from (5.15). It is also clear that μ, as defined by (5.15), is a measure. Now if Q is a q-open set such that $\mu_0(Q) < \infty$ then $\mu(Q) < \infty$ and both $\mu_0\big|_Q$ and $\mu\big|_Q$ are regular relative to the induced Euclidean topology on $\partial\Omega$. Since they agree on open sets, the regularity implies (5.16) (b).

If A is a Borel set such that $C_{2/q,q'}(A) = 0$ and Q is a q-open set then $Q \setminus A$ is q-open and consequently

$$\mu(Q) = \mu_0(Q) = \mu_0(Q \setminus A) = \mu(Q \setminus A).$$

Thus μ is essentially absolutely continuous. It is obvious by its definition that μ is outer regular with respect to $C_{2/q,q'}$. Thus $\mu \in \mathbb{M}_q(\partial\Omega)$. $\square$

5.4. The boundary trace on the regular set. First we describe some properties of moderate solutions. In this connection it is convenient to introduce a related term: a solution is *strictly moderate* if

$$|u| \leq v, \quad v \text{ harmonic}, \quad \int_\Omega v^q \rho \, dx < \infty. \tag{5.17}$$

A positive solution u is strictly moderate if and only if $\operatorname{tr} u \in W_+^{-2/q,q}(\partial\Omega)$ (see [**15**]).

Notation 5.1. Let $\Pi : \Omega_{\beta_0} \to \partial\Omega$ be the mapping given by $\Pi(x) = \sigma(x)$ (see Notation 1.1) and put $\Pi_\beta := \Pi\big|_{\Sigma_\beta}$.

1: If ϕ is a function defined on $\partial\Omega$ put $\phi^* := \phi \circ \Pi$. This function is called the *normal lifting* of ϕ to Ω_{β_0}. Similarly, if ϕ is defined on a set $Q \subset \partial\Omega$, ϕ^* is the normal lifting of ϕ to $\Omega_{\beta_0}(Q)$.

2: If φ is a function defined on Σ_β we define the *normal projection* of φ onto $\partial\Omega$ by

$$\varphi_*^\beta(\xi) = \varphi(\Pi_\beta^{-1}(\xi)), \quad \forall \xi \in \partial\Omega, \quad \Pi_\beta = \Pi\big|_{\Sigma_\beta} : \Sigma_\beta \mapsto \partial\Omega.$$

If v is a function defined on Ω_{β_0} then v_*^β denotes the normal projection of $v(\beta, \cdot)$ onto $\partial\Omega$, for $\beta \in (0, \beta_0)$.

Proposition 5.10. *Let u be a moderate solution of* (1.1)*, not necessarily positive. Then:*

(i) $u \in L^1(\Omega) \cap L^q(\Omega; \rho)$ and u possesses a boundary trace $\operatorname{tr} u$ *given by a bounded Borel measure μ which is attained in the sense of weak convergence of measures:*

$$\lim_{\beta \to 0} \int_{\Sigma_\beta} u\phi^* dS = \lim_{\beta \to 0} \int_{\partial\Omega} u_*^\beta \phi dS = \int_{\partial\Omega} \phi \, d\mu, \tag{5.18}$$

for every $\phi \in C(\partial\Omega)$.

(ii) A bounded *Borel measure μ is the boundary trace of a solution of* (1.1) *if and only if it is absolutely continuous relative to $C_{2/q,q'}$. When this is the case, there exists a sequence $\{\mu_n\} \subset W^{-2/q,q}(\partial\Omega)$ such that $\mu_n \to \mu$ in total variation norm. If μ is positive, the sequence can be chosen to be increasing. Note that these facts imply that μ is a trace if and only if $|\mu|$ is a trace.*

(iii) u is strictly moderate if and only if $|\operatorname{tr} u| \in W^{-2/q,q}(\partial\Omega)$. In this case the boundary trace is also attained in the sense of weak convergence in $W^{-2/q,q}(\partial\Omega)$ of $\{u_*^\beta : \beta \in (0, \beta_0)\}$ as $\beta \to 0$. In particular (5.18) *holds for every $\phi \in W^{2/q,q'}(\partial\Omega) \cup C(\partial\Omega)$.*

(iv) If $\mu := \operatorname{tr} u$ and $\{\mu_n\}$ is as in (ii) then $u = \lim u_{\mu_n}$. In particular, if $u > 0$ then u is the limit of an increasing sequence of strictly moderate solutions.

(v) The measure $\mu = \operatorname{tr} u$ is regular relative to the q-topology.

(vi) If u is positive (not necessarily strictly moderate), (5.18) *is valid for every $\phi \in (W_+^{2/q,q'} \cap L^\infty)(\partial\Omega)$.*

Remark. Assertions (i)-(iv) are well known. For proofs see [**15**] which also contains further relevant citations.

Proof of (v). If μ is a trace then μ_+ and μ_- are traces of solutions of (1.1). Therefore it is enough to prove (v) in the case that μ is a positive measure.

Every bounded Borel measure on $\partial\Omega$ is regular in the usual sense:

$$\begin{aligned}\mu(E) &= \inf\{\mu(O) : E \subset O,\ O \text{ relatively open}\}\\ &= \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}\end{aligned}$$

for every Borel set $E \subset \partial\Omega$. Since

$$\begin{aligned}\mu(E) &\leq \inf\{\mu(D) : E \subset D \subset \partial\Omega,\ D\ q\text{-open}\}\\ &\leq \inf\{\mu(O) : E \subset O \subset \partial\Omega,\ O \text{ relatively open}\}\end{aligned}$$

it follows that such a measure is also regular with respect to the q-topology.

Proof of (vi). By (ii) there exists an increasing sequence of strictly moderate solutions $\{v_n\}$ such that $v_n \uparrow u$. If $\mu_n := \operatorname{tr} v_n$ then

$$\lim_{\beta\to 0}\int_{\Sigma_\beta} v_n\phi^* dS = \int_{\partial\Omega}\phi\, d\mu_n,$$

for every $\phi \in W^{2/q,q'}(\partial\Omega)$. Since $\{\mu_n\}$ increases and converges weakly to $\mu = \operatorname{tr} u$ it follows that $\mu_n \to \mu$ in total variation. Hence

$$\int_{\partial\Omega}\phi\, d\mu_n \to \int_{\partial\Omega}\phi\, d\mu$$

for every bounded $\phi \in W^{2/q,q'}(\partial\Omega)$. If, in addition, $\phi \geq 0$, we obtain

$$\liminf_{\beta\to 0}\int_{\Sigma_\beta} u\phi^* dS \geq \int_{\partial\Omega}\phi\, d\mu.$$

On the other hand, since $u^*_\beta \to \mu$ in the sense of weak convergence of measures, it follows that

$$\limsup \int_E u^*_\beta dS \leq \mu, \quad \liminf \int_A u^*_\beta dS \geq \mu \tag{5.19}$$

for any closed set $E \subset \partial\Omega$, respectively, open set $A \subset \partial\Omega$). It is easily seen that, in our case, this extends to any q-closed set E (resp. q-open set A). Therefore if A is q-open and

$$\mu(A) = \mu(\tilde{A}),$$

then

$$\lim \int_A u^\beta_* dS = \mu(A). \tag{5.20}$$

If $\phi \in W^{2/q,q'}(\partial\Omega)\cap L^\infty(\partial\Omega)$ and $I \subset \mathbb{R}$ is a bounded open interval then, by [**1**, Prop. 6.1.2, Prop. 6.4.10], $A := \phi^{-1}(I)$ is quasi open. Without loss of generality we may assume that $\phi \leq 1$. Given $k \in \mathbb{N}$ and $m = 0, \dots, 2^k - 1$ choose a number $a_{m,k}$ in the interval $(m2^{-k}, (m+1)2^{-k})$ such that $\mu_{\mathcal{R}}(\phi^{-1}(\{a_{m,k}\}) = 0$. Put

$$A_{m,k} = \phi^{-1}((a_{m,k}, a_{m+1,k}]) \quad m = 1, \dots, 2^k - 1, \quad A_{0,k} = \phi^{-1}((a_{0,k}, a_{1,k}])$$

and

$$f_k = \sum_{m=0}^{2^k-1} m2^{-k}\chi_{A_{m,k}}.$$

Then $f_k \to \phi$ uniformly and, by (5.20),

$$\lim_{\beta\to 0}\int_{\partial\Omega} f_k u^\beta_* dS = \int_{\partial\Omega} f_k d\mu_{\mathcal{R}}.$$

This implies assertion (vi). $\square$

THEOREM 5.11. *Let $u \in \mathcal{U}(\Omega)$.*

(i) *The regular set $\mathcal{R}(u)$ is σ-regular and consequently it has a regular decomposition $\{Q_n\}$.*

(ii) *Let*

$$v_{\mathcal{R}} := \sup\{[u]_Q : Q \text{ q-open and regular }\}. \tag{5.21}$$

Then there exists an increasing sequence of moderate solutions $\{w_n\}$ such that

$$\operatorname{supp}^q_{\partial\Omega} w_n \overset{q}{\subset} \mathcal{R}(u), \quad w_n \uparrow v_{\mathcal{R}}. \tag{5.22}$$

(Thus $v_{\mathcal{R}}$ is σ-moderate.)

(iii) *Let F be a q-closed set such that $F \overset{q}{\subset} \mathcal{R}(u)$. Then, for every $\epsilon > 0$, there exists a q-open regular set Q_ϵ such that $C_{2/q,q'}(F \setminus Q_\epsilon) < \epsilon$. If, in addition, $[u]_F$ is moderate then F is regular; consequently there exists a q-open regular set Q such that $F \overset{q}{\subset} Q$.*

(iv) *With $\{Q_n\}$ as in (i), denote*

$$v_n := [u]_{Q_n}, \quad \mu_n := \operatorname{tr} v_n, \quad v := \lim v_n, \quad \mu_{\mathcal{R}} := \lim \mu_n. \tag{5.23}$$

Then,

$$v = v_{\mathcal{R}}. \tag{5.24}$$

Furthermore, for every q-open regular set Q,

$$\mu_{\mathcal{R}} \chi_{\tilde{Q}} = \operatorname{tr}[u]_Q = \operatorname{tr}[v_{\mathcal{R}}]_Q. \tag{5.25}$$

Finally, $\mu_{\mathcal{R}}$ is q-locally finite on $\mathcal{R}(u)$ and σ-finite on $\mathcal{R}_0(u) := \cup Q_n$.

(v) *If $\{w_n\}$ is a sequence of moderate solutions satisfying conditions* (5.22) *then,*

$$\mu_{\mathcal{R}} = \lim \operatorname{tr} w_n \tag{5.26}$$

(vi) *The* regularized measure *$\bar{\mu}_{\mathcal{R}}$ given by*

$$\bar{\mu}_{\mathcal{R}}(E) := \inf\{\mu_{\mathcal{R}}(Q) : E \subset Q, \quad Q \text{ q-open} \quad \forall E \subset \partial\Omega, \ E \text{ Borel}\} \tag{5.27}$$

is q-perfect.

(vii) $u \underset{\mathcal{R}(u)}{\approx} v_{\mathcal{R}}$.

(viii) *For every q-closed set $F \overset{q}{\subset} \mathcal{R}(u)$:*

$$[u]_F = [v_{\mathcal{R}}]_F. \tag{5.28}$$

If, in addition, $\mu_{\mathcal{R}}(F) < \infty$ then $[u]_F$ is moderate and

$$\operatorname{tr}[u]_F = \mu_{\mathcal{R}} \chi_F. \tag{5.29}$$

(ix) *If F is a q-closed set then*

$$\mu_{\mathcal{R}}(F) < \infty \iff [u]_F \text{ is moderate} \iff F \text{ is regular}. \tag{5.30}$$

PROOF. (i) By [**1**, Sec. 6.5.11] the (α, p)-fine topology possesses the quasi-Lindelöf property. This implies that $\mathcal{R}(u)$ is σ-regular. By Lemma 5.4 $\mathcal{R}(u)$ has a regular decomposition $\{Q_n\}$. Recall that $\tilde{Q}_n \subset Q_{n+1}$ and $C_{2/q,q'}(\mathcal{R}(u) \setminus \mathcal{R}_0(u)) = 0$.

(ii) This assertion is an immediate consequence of (5.21) and Proposition 3.3.

(iii) By definition, every point in $\mathcal{R}(u)$ possesses a q-open regular neighborhood. Therefore, the existence of a set Q_ϵ, as in the first part of this assertion, is an

immediate consequence of Lemma 2.5. Let O_ϵ be an open set containing $F \setminus Q_\epsilon$ such that $C_{2/q,q'}(O_\epsilon) < 2\epsilon$. Put $F_\epsilon := F \setminus O_\epsilon$. Then F_ϵ is a q-closed set, $F_\epsilon \subset F$, $C_{2/q,q'}(F \setminus F_\epsilon) < 2\epsilon$ and $F_\epsilon \overset{q}{\subset} Q_\epsilon$.

Assertion 1. *Let E be a q-closed set, D a q-open regular set and $E \overset{q}{\subset} D$. Then there exists a decreasing sequence of q-open sets $\{G_n\}_{n=1}^\infty$ such that*

$$E \overset{q}{\subset} G_{n+1} \subset \tilde{G}_{n+1} \overset{q}{\subset} G_n \overset{q}{\subset} D \tag{5.31}$$

and

$$[u]_{G_n} \to [u]_E \quad in \quad L^q(\Omega, \rho). \tag{5.32}$$

By Lemma 2.6 and Theorem 4.4, there exists a decreasing sequence of q-open sets $\{G_n\}$ satisfying (5.31) and, in addition, such that $[u]_{G_n} \downarrow [u]_E$ locally uniformly in Ω. Since $[u]_{G_n} \le [u]_D$ and the latter is a moderate solution we obtain (5.32).

Put

$$E_n := \cup_{m=1}^n F_{1/m}, \quad D_n := \cup_{m=1}^n Q_{1/m}.$$

Then E_n is q-closed, D_n is q-open and regular and $E_n \overset{q}{\subset} D_n$. Therefore, by Assertion 1, it is possible to choose a sequence of q-open regular sets $\{V_n\}$ such that

$$E_n \overset{q}{\subset} V_n \subset \tilde{V}_n \overset{q}{\subset} D_n, \quad \left\| [u]_{V_n} - [u]_{E_n} \right\|_{L^q(\Omega,\rho)} \le 2^{-n}. \tag{5.33}$$

By Theorem 4.4,

$$[u]_F \le [u]_{E_n} + [u]_{F \setminus E_n} \quad \text{and} \quad [u]_{F \setminus E_n} \downarrow 0.$$

Therefore $[u]_{E_n} \uparrow [u]_F$.

If, in addition, $[u]_F$ is moderate then

$$[u]_{E_n} \uparrow [u]_F \quad \text{in} \quad L^q(\Omega, \rho)$$

and consequently, by (5.33),

$$[u]_{V_n} \to [u]_F \quad \text{in} \quad L^q(\Omega, \rho).$$

Let $\{V_{n_k}\}$ be a subsequence such that

$$\left\| [u]_{V_{n_k}} - [u]_F \right\|_{L^q(\Omega,\rho)} \le 2^{-k}. \tag{5.34}$$

Recall that $E_n \overset{q}{\subset} V_n \cap F$ and that $C_{2/q,q'}(F \setminus E_n) \downarrow 0$. Therefore $C_{2/q,q'}(F \setminus V_n) \to 0$. Consequently $F \overset{q}{\subset} W := \cap_{k=1}^\infty V_{n_k}$ and, in view of (5.34), $[u]_W$ is moderate. Obviously this implies that W is pre-regular (any q-closed set $E \overset{q}{\subset} W$ has the property that $[u]_E$ is moderate) and F is regular. Finally, by Lemma 2.4, every q-closed regular set is contained in a q-open regular set.

(iv) Let Q be a q-open regular set and put $\mu_Q = \operatorname{tr}[u]_Q$. If F is a q-closed set such that $F \overset{q}{\subset} Q$ then, by Theorem 5.5,

$$[u]_F = \mu_Q \chi_F. \tag{5.35}$$

In particular the compatibility condition holds: if Q, Q' are q-open regular sets then

$$\mu_{Q \cap Q'} = \mu_Q \chi_{\tilde{Q} \cap \tilde{Q}'} = \mu_{Q'} \chi_{\tilde{Q} \cap \tilde{Q}'}. \tag{5.36}$$

With the notation of (5.23), $[v_{n+k}]_{Q_k} = v_k$ and hence $\mu_{n+k}\chi_{\tilde{Q}_k} = \mu_k$ for every $k \in \mathbb{N}$.

Let F be an arbitrary q-closed subset of $\mathcal{R}(u)$. Since $C_{2/q,q'}(F \setminus Q_n) \to 0$ it follows that

$$[v_n]_F = [u]_{F\cap\tilde{Q}_n} \uparrow [u]_F. \tag{5.37}$$

In addition, $[v]_F \geq \lim[v_n]_F = [u]_F$ and $v \leq u$ lead to,

$$[u]_F = [v]_F. \tag{5.38}$$

If Q is a q-open regular set, $[u]_Q = \lim[v_n]_Q \leq \lim v_n =: v$ and so $v_{\mathcal{R}} \leq v$. On the other hand it is obvious that $v \leq v_{\mathcal{R}}$. Thus (5.24) holds.

By (5.35) and (5.37), if F is a q-closed subset of $\mathcal{R}(u)$ and $[u]_F$ is moderate,

$$\mathrm{tr}\,[u]_F = \lim \mathrm{tr}\,[v_n]_F = \lim \mu_n\chi_F = \mu_{\mathcal{R}}\chi_F, \tag{5.39}$$

which implies (5.25). This also shows that $\mu_{\mathcal{R}}\chi_F$ is independent of the choice of the sequence $\{\mu_n\}$ used in its definition. This remains valid for any q-closed set $F \overset{q}{\subset} \mathcal{R}(u)$ because $C_{2/q,q'}(\mathcal{R}(u)\setminus\mathcal{R}_0(u)) = 0$ and $\mu_{\mathcal{R}}$ is σ-finite on $\mathcal{R}_0(u)$. The last observation is a consequence of the fact that $\mathcal{R}_0(u)$ has a regular decomposition.

The assertion that $\mu_{\mathcal{R}}$ is q-locally finite on $\mathcal{R}(u)$ is a consequence of the fact that every point in $\mathcal{R}(u)$ is contained in a q-open regular set.

(v) If w is a moderate solution and $w \leq v_{\mathcal{R}}$ and $\mathrm{supp}^q_{\partial\Omega}\, w \overset{q}{\subset} \mathcal{R}(u)$ then $\tau := \mathrm{tr}\, w \leq \mu_{\mathcal{R}}$. Indeed

$$[w]_{Q_n} \leq [v_{\mathcal{R}}]_{Q_n} = v_n,\ [w]_{Q_n} \uparrow w \Longrightarrow \mathrm{tr}\,[w]_{Q_n} \uparrow \tau \leq \lim \mathrm{tr}\, v_n = \mu_{\mathcal{R}}.$$

Now, let $\{w_n\}$ be an increasing sequence of moderate solutions such that $F_n := \mathrm{supp}^q_{\partial\Omega}\, w_n \overset{q}{\subset} \mathcal{R}(u)$ and $w_n \uparrow v_{\mathcal{R}}$. We must show that, if $\nu_n := \mathrm{tr}\, w_n$,

$$\nu := \lim \nu_n = \mu_{\mathcal{R}}. \tag{5.40}$$

By the previous argument $\nu \leq \mu_{\mathcal{R}}$. The opposite inequality is obtained as follows. Let D be a q-open regular set and let K be a compact subset of D such that $C_{2/q,q'}(K) > 0$.

$$w_n \leq [w_n]_D + [w_n]_{D^c} \Longrightarrow v_{\mathcal{R}} = \lim w_n \leq \lim[w_n]_D + U_{D^c}.$$

The sequence $\{[w_n]_D\}$ is dominated by the moderate function $[v_{\mathcal{R}}]_D$. In addition $\mathrm{tr}\,[w_n]_D = \nu_n\chi_{\tilde{D}} \uparrow \nu\chi_{\tilde{D}}$. Hence, $\nu\chi_{\tilde{D}}$ is a bounded measure and $[w_n]_D \uparrow u_{\nu\chi_{\tilde{D}}}$ where the function on the right is the moderate solution with trace $\nu\chi_{\tilde{D}}$. Consequently

$$v_{\mathcal{R}} = \lim w_n \leq u_{\nu\chi_{\tilde{D}}} + U_{D^c}.$$

This in turn implies

$$([v_{\mathcal{R}}]_K - u_{\nu\chi_{\tilde{D}}})_+ \leq \inf(U_{D^c}, U_K)$$

the function on the left being a subsolution and the one on the left a supersolution. Therefore

$$([v_{\mathcal{R}}]_K - u_{\nu\chi_{\tilde{D}}})_+ \leq [[U]_{D^c}]_K = 0.$$

Thus, $[v_{\mathcal{R}}]_K \leq u_{\nu\chi_{\tilde{D}}}$ and hence $\mu_{\mathcal{R}}\chi_K \leq \nu\chi_{\tilde{D}}$. Further, if Q is a q-open set such that $\tilde{Q} \overset{q}{\subset} D$ then, in view of the fact that

$$\sup\{\mu_{\mathcal{R}}\chi_K : K \in Q,\ K \text{ compact}\} = \mu\chi_Q,$$

we obtain,

$$\mu_{\mathcal{R}}\chi_{Q} \leq \nu\chi_{\tilde{D}}. \tag{5.41}$$

Applying this inequality to the sets Q_m, Q_{m+1} we finally obtain

$$\mu_{\mathcal{R}}\chi_{Q_m} \leq \nu\chi_{\tilde{Q}_{m+1}} \leq \nu\chi_{Q_{m+2}}.$$

Letting $m \to \infty$ we conclude that $\mu_{\mathcal{R}} \leq \nu$. This completes the proof of (5.40) and of assertion (v).

(vi) The measure $\mu_{\mathcal{R}}$ is essentially absolutely continuous relative to $C_{2/q,q'}$ (see Definition 5.8). Therefore this assertion follows from Lemma 5.9.

(vii) By (4.5)

$$u \leq [u]_{Q_n} + [u]_{\partial\Omega\setminus Q_n}.$$

By Theorem 3.9(c)

$$[u]_{\partial\Omega\setminus Q_n} \downarrow [u]_{\partial\Omega\setminus\mathcal{R}_0}.$$

Hence

$$\lim(u - [u]_{Q_n}) = u - v_{\mathcal{R}} \leq [u]_{\partial\Omega\setminus\mathcal{R}_0}$$

so that $u \ominus v_{\mathcal{R}} \underset{\mathcal{R}_0}{\approx} 0$. Since $v_{\mathcal{R}} \leq u$ this is equivalent to the statement $u \underset{\mathcal{R}}{\approx} v_{\mathcal{R}}$.

(viii) (5.28) was established before, see (5.38). Alternatively, it follows from the previous assertion and Theorem 4.7. If $\mu_{\mathcal{R}}(F)$ is finite then (5.29) is a consequence of (i) and (5.35). Indeed, $F_n := F\cap\tilde{Q}_n \uparrow F_0 \overset{q}{\sim} F$. Hence, $[u]_F \leq [u]_{F_n} + [u]_{F\setminus F_n}$ and $C_{2/q,q'}(F \setminus F_n) \downarrow 0$. Hence $[u]_F = \lim[u]_{F_n}$ and $\operatorname{tr}[u]_{F_n} = \mu_{\mathcal{R}}\chi_{F_n} \uparrow \mu_{\mathcal{R}}\chi_{F_0} = \mu_{\mathcal{R}}\chi_F$. Since $\mu_{\mathcal{R}}\chi_F$ is a bounded measure, $[u]_F$ is moderate and (5.29) holds.

(ix) If $\mu_{\mathcal{R}}(F) < \infty$ then, by (viii), $[u]_F$ is moderate and, by (iii), F is regular. Conversely, if F is regular then $[u]_F$ is moderate and, by (5.25), $\mu_{\mathcal{R}}(F) < \infty$. □

5.5. The precise boundary trace.

DEFINITION 5.12. Let $q_c \leq q$ and $u \in \mathcal{U}(\Omega)$.

- **a:** The solution $v_{\mathcal{R}}$ defined by (5.21) is called the *regular component* of u and will be denoted by u_{reg}.
- **b:** Let $\{v_n\}$ be an increasing sequence of moderate solutions satisfying condition (5.22) and put $\mu_{\mathcal{R}} = \mu_{\mathcal{R}}(u) := \lim \operatorname{tr} v_n$. Then, the regularized measure $\bar{\mu}_{\mathcal{R}}$, defined by (5.27), is called the *regular boundary trace* of u. It will be denoted by $\operatorname{tr}_{\mathcal{R}} u$.
- **c:** The couple $(\operatorname{tr}_{\mathcal{R}} u, \mathcal{S}(u))$ is called the *precise boundary trace* of u and will be denoted by $\operatorname{tr}^c u$.
- **d:** Let ν be the Borel measure on $\partial\Omega$ given by

$$\nu(E) = \begin{cases} (\operatorname{tr}_{\mathcal{R}} u)(E) & \text{if } E \subset \mathcal{R}(u), \\ \nu(E) = \infty & \text{if } E \cap \mathcal{S}(u) \neq \emptyset, \end{cases} \tag{5.42}$$

for every Borel set $E \subset \partial\Omega$. Then ν is the measure representation of the precise boundary trace of u, to be denoted by $\operatorname{tr} u$.

Note that, by Theorem 5.11 (v), the measure $\mu_{\mathcal{R}}$ is independent of the choice of the sequence $\{v_n\}$.

THEOREM 5.13. *Assume that $u \in \mathcal{U}(\Omega)$ is a σ-moderate solution, i.e., there exists an increasing sequence $\{u_n\}$ of positive moderate solutions such that $u_n \to u$. Let $\mu_n := \operatorname{tr} u_n$, $\mu_0 := \lim \mu_n$ and put*

$$\mu(E) := \inf\{\mu_0(D) : E \subset D \subset \partial\Omega,\ D\ q\text{-open}\}, \tag{5.43}$$

for every Borel set $E \subset \partial\Omega$. Then:

(i) *μ is the precise boundary trace of u and μ is q-perfect. In particular μ is independent of the sequence $\{u_n\}$ which appears in its definition.*

(ii) *If A is a Borel set such that $\mu(A) < \infty$ then $\mu(A) = \mu_0(A)$.*

(iii) *A solution $u \in \mathcal{U}(\Omega)$ is σ-moderate if and only if*

$$u = \sup\{v \in \mathcal{U}(\Omega) : v \text{ moderate } \quad v \le u\}, \tag{5.44}$$

which is equivalent to

$$u = \sup\{u_\tau : \tau \in W_+^{-2/q,q}(\partial\Omega), \quad \tau \le \operatorname{tr} u\}. \tag{5.45}$$

Thus, if u is σ-moderate, there exists an increasing sequence of strictly moderate *solutions converging to u.*

(iv) *If u, w are σ-moderate solutions,*

$$\operatorname{tr} w \le \operatorname{tr} u \iff w \le u. \tag{5.46}$$

PROOF. (i) Let Q be a q-open set and A a Borel set such that $C_{2/q,q'}(A) = 0$. Then $\mu_n(A) = 0$ so that $\mu_0(A) = 0$. Therefore $\mu_0(Q \setminus A) = \mu_0(Q)$. Thus μ_0 is essentially absolutely continuous and, by Lemma 5.9, μ is q-perfect.

Let $\{D_n\}$ be a regular decomposition of $\mathcal{R}(u)$. Put $D'_n = \mathcal{R}(u) \setminus D_n$ and observe that $D'_n \downarrow E$ where $C_{2/q,q'}(E) = 0$. Therefore

$$u_{\mu_n \chi_{D'_n}} \downarrow 0.$$

Since,

$$\mu_n \chi_{\mathcal{R}(u)} = \mu_n \chi_{D_n} + \mu_n \chi_{D'_n},$$

it follows that

$$\left| u_{\mu_n \chi_{\mathcal{R}(u)}} - \lim u_{\mu_n \chi_{D_n}} \right| \le u_{\mu_n \chi_{D'_n}} \to 0.$$

Now

$$u_n \le u_{\mu_n \chi_{\mathcal{R}(u)}} + [u]_{\mathcal{S}(u)}.$$

Hence

$$u - [u]_{\mathcal{S}(u)} \le w := \lim u_{\mu_n \chi_{\mathcal{R}(u)}} = \lim u_{\mu_n \chi_{D_n}} \le u_{reg}.$$

This implies $u \ominus [u]_{\mathcal{S}(u)} \le u_{reg}$. But the definition of $u_{reg} (= v_{\mathcal{R}})$ implies that $u_{reg} \le u \ominus [u]_{\mathcal{S}(u)}$. Therefore $\lim u_{\mu_n \chi_{D_n}} = u_{reg}$. Thus the sequence $\{u_{\mu_n \chi_{D_n}}\}$ satisfies condition (5.22) and consequently, by Theorem 5.11 (iv) and Definition 5.12,

$$\lim \mu_n \chi_{D_n} = \mu_{\mathcal{R}}, \quad \operatorname{tr}_{\mathcal{R}} u = \bar{\mu}_{\mathcal{R}}. \tag{5.47}$$

If $\xi \in \mathcal{S}(u)$ then, for every q-open neighborhood Q of ξ, $\int_\Omega [u]_Q^q \rho dx = \infty$. This implies: $\mu_n(\tilde{Q}) \to \infty$. To verify this fact, assume that, on the contrary, there exists a subsequence (still denoted $\{\mu_n\}$) such that $\sup \mu_n(\tilde{Q}) < \infty$. Denote by $v_{n,Q}$ and $w_{n,Q}$ the solutions with boundary trace $\chi_Q \operatorname{tr} u_n$ and $\chi_{Q^c} \operatorname{tr} u_n$ respectively. Then

$$u_n \le v_{n,Q} + w_{n,Q} \Longrightarrow u \le v_Q + w_Q, \quad v_Q = \lim v_{n,Q}, \quad w_Q = \lim w_{n,Q}.$$

Then v_Q is moderate and w_Q vanishes in Q. If D is a q-open set such that $\xi \in D \subset \tilde{D} \subset Q$ then $[w_Q]_D = 0$. Therefore

$$\min(u, U_D) \leq \min(v_Q, U_D) + \min(w_Q, U_D) \Longrightarrow [u]_D \leq [v_Q]_D,$$

which brings us to a contradiction. In conclusion, if $\xi \in \mathcal{S}(u)$ then $\mu_0(\tilde{Q}) = \infty$ for every q-open neighborhood of ξ. Consequently $\mu(\xi) = \infty$. This fact and (5.47) imply that μ is the precise trace of u.

(ii) If $\mu(A) < \infty$ then A is contained in a q-open set D such that $\mu_0(D) < \infty$ and, by Lemma 5.9, $\mu(A) = \mu_0(A)$.

(iii) Let $u \in \mathcal{U}(\Omega)$ be σ-moderate and put

$$u^* := \sup\{v \in \mathcal{U}(\Omega) : v \text{ moderate} \quad v \leq u\}. \tag{5.48}$$

By its definition $u^* \leq u$. On the other hand, since there exists an increasing sequence of moderate solutions $\{u_n\}$ converging to u, it follows that $u \leq u^*$. Thus $u = u^*$.

Conversely, if $u \in \mathcal{U}(\Omega)$ and $u = u^*$ then, by Proposition 3.3, there exists an increasing sequence of moderate solutions $\{u_n\}$ converging to u. Therefore u is σ-moderate.

In view of Proposition 5.10 (iv),

$$u^* \leq \sup\{u_\tau : \tau \in W_+^{-2/q,q}(\partial\Omega), \quad \tau \leq \operatorname{tr} u\} =: u^\ddagger.$$

On the other hand, if u is σ-moderate, $\tau \in W_+^{-2/q,q}(\partial\Omega)$ and $\tau \leq \operatorname{tr} u$ then (with μ_n and u_n as in the statement of the theorem), $\operatorname{tr}(u_\tau \ominus u_n) = (\tau - \mu_n)_+ \downarrow 0$. Hence $u_\tau \ominus u_n \downarrow 0$, which implies that $u_\tau \ominus u = 0$, i.e. $u_\tau \leq u$. Therefore $u \geq u^\ddagger$. Thus (5.44) implies (5.45) and each of them implies that u is σ-moderate. Therefore the two are equivalent.

(iv) The assertion $\Longrightarrow$ is a consequence of (5.45). To verify the assertion $\Longleftarrow$ it is sufficient to show that if w is moderate, u is σ-moderate and $w \leq u$ then $\operatorname{tr} w \leq \operatorname{tr} u$. Let $\{u_n\}$ be an increasing sequence of positive moderate solutions converging to u. Then $u_n \vee w \leq u$ and consequently $u_n \leq u_n \vee w \uparrow u$. Therefore $\operatorname{tr}(u_n \vee w) \uparrow \mu' \leq \operatorname{tr} u$ so that $\operatorname{tr} w \leq \operatorname{tr} u$.

□

THEOREM 5.14. *Let $u \in \mathcal{U}(\Omega)$ and put $\nu = \operatorname{tr} u$.*

(i) *u_{reg} is σ-moderate and $\operatorname{tr} u_{reg} = \operatorname{tr}_{\mathcal{R}} u$.*

(ii) *If $v \in \mathcal{U}(\Omega)$*

$$v \leq u \Longrightarrow \operatorname{tr} v \leq \operatorname{tr} u. \tag{5.49}$$

If F is a q-closed set then

$$\operatorname{tr}[u]_F \leq \nu \chi_F. \tag{5.50}$$

(iii) *A singular point can be characterized in terms of the measure ν as follows:*

$$\xi \in \mathcal{S}(u) \iff \nu(Q) = \infty \quad \forall Q : \ \xi \in Q, \ Q \text{ q-open}. \tag{5.51}$$

(iv) *If Q is a q-open set then:*

$$Q \text{ pre-regular} \iff \nu(F \cap Q) < \infty \quad \forall F \overset{q}{\subset} Q : \ F \ q\text{-closed} \tag{5.52}$$

$$Q \text{ regular} \iff \exists \text{ Borel set } A\text{: } C_{2/q,q'}(A) = 0, \ \nu(\tilde{Q} \setminus A) < \infty. \tag{5.53}$$

(v) *The singular set of* $u_{_{reg}}$ *may not be empty. In fact*

$$\mathcal{S}(u)\setminus b_q(\mathcal{S}(u))\subset\mathcal{S}(u_{_{reg}})\subset\mathcal{S}(u)\cap\widetilde{\mathcal{R}(u)},\tag{5.54}$$

where $b_q(\mathcal{S}(u))$ *is the set of* $C_{2/q,q'}$*-thick points of* $\mathcal{S}(u)$*, (see Notation 2.1).*

(vi) *Put*

$$\mathcal{S}_0(u):=\{\xi\in\partial\Omega:\ \nu(Q\setminus\mathcal{S}(u))=\infty\quad\forall Q:\ \xi\in Q,\ Q\ q\text{-open}\}.\tag{5.55}$$

Then

$$\mathcal{S}(u_{_{reg}})\setminus b_q(\mathcal{S}(u))\subset\mathcal{S}_0(u)\subset\mathcal{S}(u_{_{reg}})\cup b_q(\mathcal{S}(u))\tag{5.56}$$

Remark. This result complements Theorem 5.11 which deals with the regular boundary trace.

Proof. (i) By Theorem 5.11 (ii) $u_{_{reg}}$ is σ-moderate. The second part of the statement follows from Definition 5.12 and Theorem 5.13 (i).

(ii) If $v\le u$ then $\mathcal{R}(u)\subset\mathcal{R}(v)$ and by definition $v_{_{reg}}\le u_{_{reg}}$. By Theorem 5.13 (iv) $\operatorname{tr}v_{_{reg}}\le\operatorname{tr}u_{_{reg}}$ and consequently $\operatorname{tr}v\le\operatorname{tr}u$. (5.50) is an immediate consequence of (5.49).

(iii) If $\xi\in\mathcal{R}(u)$ there exists a q-open regular neighborhood Q of ξ. Hence $\nu(Q)=\operatorname{tr}_{\mathcal{R}}u(Q)<\infty$. If $\xi\in\mathcal{S}(u)$, it follows immediately from the definition of precise trace that $\nu(Q)=\infty$ for every q-open neighborhood Q of ξ.

(iv) If Q is pre regular then $[u]_F$ is moderate for every q-closed set $F\overset{q}{\subset}Q$ and $Q\overset{q}{\subset}\mathcal{R}(u)$. By Theorem 5.11 (viii) this implies: $\operatorname{tr}[u]_F=(\mu_{\mathcal{R}}(u))\chi_{_F}$ and consequently $\nu(F\cap Q)=(\mu_{\mathcal{R}}(u))(F\cap Q)=(\mu_{\mathcal{R}}(u))(F)<\infty$. Therefore (5.52) holds in the direction $\Longrightarrow$.

Conversely, if Q is a q-open set, F a q-closed set, $F\overset{q}{\subset}Q$ and $\nu(F\cap Q)<\infty$ then, by Definition 5.12, $F\cap Q\subset\mathcal{R}(u)$ which implies $F\overset{q}{\subset}\mathcal{R}(u)$ and $\mu_{\mathcal{R}}(u)(F)<\infty$. Therefore, by Theorem 5.11 (viii) $[u]_F$ is moderate. This implies (5.52) in the opposite direction.

If Q is regular there exists a pre-regular set D such that $\tilde{Q}\overset{q}{\subset}D$. Therefore (5.52) implies (5.53) in the direction $\Longrightarrow$. On the other hand,

$$\nu(\tilde{Q}\setminus A)<\infty\Longrightarrow\tilde{Q}\overset{q}{\subset}\mathcal{R}(u)$$

and $\mu_{\mathcal{R}}(\tilde{Q})=\mu_{\mathcal{R}}(\tilde{Q}\setminus A)<\infty$. Hence, by Theorem 5.11 (ix), $[u]_Q$ is moderate and $\tilde{Q}$ is regular.

(v) Since $\operatorname{supp}^q_{\partial\Omega}u_{_{reg}}\subset\widetilde{\mathcal{R}(u)}$ and $\mathcal{R}(u)\subset\mathcal{R}(u_{_{reg}})$ we have

$$\mathcal{S}(u_{_{reg}})\subset\widetilde{\mathcal{R}(u)}\cap\mathcal{S}(u).$$

Next we show that $\mathcal{S}(u)\setminus b_q(\mathcal{S}(u))\subset\mathcal{S}(u_{_{reg}})$.

If $\xi\in\mathcal{S}(u)\setminus b_q(\mathcal{S}(u))$ then $\mathcal{R}(u)\cup\{\xi\}$ is a q-open neighborhood of ξ. By (i) $u_{_{reg}}$ is σ-moderate and consequently (by Theorem 5.13 (i)) its trace is q-perfect. Therefore, if Q_0 is a q-open neighborhood of ξ and $Q=Q_0\cap(\{\xi\}\cup\mathcal{R}(u))$ then

$$(\operatorname{tr}u_{_{reg}})(Q)=(\operatorname{tr}u_{_{reg}})(Q\setminus\{\xi\})=(\operatorname{tr}u)(Q\setminus\{\xi\}).$$

The last equality is valid because $Q\setminus\{\xi\}\subset\mathcal{R}(u)$. Let D be a q-open set such that $\xi\in D\subset\tilde{D}\subset Q$. If $\operatorname{tr}u(\tilde{D}\setminus\{\xi\})<\infty$ then, by (iv), D is regular and $\xi\in\mathcal{R}(u)$, contrary to our assumption. Therefore $\operatorname{tr}u(\tilde{D}\setminus\{\xi\})=\infty$ so that $\operatorname{tr}u_{_{reg}}(Q_0\setminus\{\xi\})=$

∞ for every q-open neighborhood Q_0 of ξ, which implies $\xi \in \mathcal{S}(u_{reg})$. This completes the proof of (5.54).

(vi) If $\xi \notin b_q(\mathcal{S}(u))$, there exists a q-open neighborhood D of ξ such that $(D \setminus \{\xi\}) \cap \mathcal{S}(u) = \emptyset$ and consequently

$$(\operatorname{tr} u_{reg})(D \setminus \{\xi\}) = (\operatorname{tr} u_{reg})(D \setminus \mathcal{S}(u)) = (\operatorname{tr} u)(D \setminus \mathcal{S}(u)). \tag{5.57}$$

If, in addition, $\xi \in \mathcal{S}_0(u)$ then

$$(\operatorname{tr} u)(D \setminus \mathcal{S}(u)) = (\operatorname{tr} u_{reg})(D \setminus \{\xi\}) = \infty.$$

If Q is an arbitrary q-open neighborhood of ξ then the same holds if D is replaced by $Q \cap D$. Therefore $(\operatorname{tr} u_{reg})(Q \setminus \{\xi\}) = \infty$ for any such Q. Consequently $\xi \in \mathcal{S}(u_{reg})$, which proves that $\mathcal{S}_0(u) \setminus b_q(\mathcal{S}(u)) \subset \mathcal{S}(u_{reg})$.

On the other hand, if $\xi \in \mathcal{S}(u_{reg}) \setminus b_q(\mathcal{S}(u))$ then there exists a q-open neighborhood D such that (5.57) holds and $(\operatorname{tr} u_{reg})(D) = \infty$. Since u_{reg} is σ-moderate, $(\operatorname{tr} u_{reg})$ is q-perfect so that $(\operatorname{tr} u_{reg})(D) = (\operatorname{tr} u_{reg})(D \setminus \{\xi\}) = \infty$. Consequently, by (5.57), $(\operatorname{tr} u)(D \setminus \mathcal{S}(u)) = \infty$. If Q is any q-open neighborhood of ξ then D can be replaced by $D \cap Q$. Consequently $(\operatorname{tr} u)(Q \setminus \mathcal{S}(u)) = \infty$ and we conclude that $\xi \in \mathcal{S}_0(u)$. This completes the proof of (5.56).

□

LEMMA 5.15. *Let $F \subset \partial\Omega$ be a q-closed set. Then $\mathcal{S}(U_F) = b_q(F)$.*

PROOF. Let ξ be a point on $\partial\Omega$ such that F is $C_{2/q,q'}$-thin at ξ. Let Q be a q-open neighborhood of ξ such that $\tilde{Q} \overset{q}{\subset} F^c$. Then $[U_F]_Q = U_{F\cap\tilde{Q}} = 0$. Therefore $\xi \in \mathcal{R}(U_F)$.

Conversely, assume that $\xi \in F \cap \mathcal{R}(U_F)$. By Theorem 5.7 there exists a q-open neighborhood Q of ξ such that $u^Q = \lim_{\beta\to 0} u^Q_\beta$ exists and u^Q is a moderate solution. Let D be a q-open neighborhood of ξ such that $\tilde{D} \overset{q}{\subset} Q$. Then Lemma 4.5 implies that $[u]_D$ is moderate so that $D \subset \mathcal{R}(u)$. In turn this implies that $C_{2/q,q'}(F \cap D) = 0$ and consequently F is q-thin at ξ. □

5.6. The boundary value problem.

Notation 5.2.

- a: Denote by $\mathfrak{M}_+(\partial\Omega)$ the space of positive Borel measures on $\partial\Omega$ (not necessarily bounded).
- b: Denote by $\mathfrak{C}_q(\partial\Omega)$ the space of couples (τ, F) such that $F \subset \partial\Omega$ is q-closed, $\tau \in \mathfrak{M}_+(\partial\Omega)$, q-supp $\tau \subset \widetilde{\partial\Omega \setminus F}$ and $\tau\chi_{\partial\Omega\setminus F}$ is q-locally finite.
- c: Denote by $\mathbb{T} : \mathfrak{C}_q(\partial\Omega) \mapsto \mathfrak{M}_+(\partial\Omega)$ the mapping given by $\nu = \mathbb{T}(\tau, F)$ where ν is defined as in (5.42) with $\mathcal{S}(u), \mathcal{R}(u)$ replaced by F, F^c respectively. ν is the measure representation of the couple (τ, F).
- d: If $(\tau, F) \in \mathfrak{C}_q(\partial\Omega)$ the set

$$F_\tau = \{\xi \in \partial\Omega : \tau(Q \setminus F) = \infty \quad \forall Q \;\; q\text{-open neighborhood of } \xi \;\}\tag{5.58}$$

is called the set of explosion points of τ.

Remark. Note that $F_\tau \subset F$ (because $\tau\chi_{\partial\Omega\setminus F}$ is q-locally finite) and $F_\tau \subset \widetilde{\partial\Omega \setminus F}$ (because τ vanishes outside this set). Thus

$$F_\tau \subset b_q(\partial\Omega \setminus F) \cap F. \tag{5.59}$$

THEOREM 5.16. *Let ν be a positive Borel measure on $\partial\Omega$.*

(i) *The boundary value problem*

$$-\Delta u + u^q = 0, \quad u > 0 \text{ in } \Omega, \quad \operatorname{tr}(u) = \nu \text{ on } \partial\Omega \tag{5.60}$$

possesses a solution if and only if $\nu \in \mathbb{M}_q(\partial\Omega)$.

(ii) *Let $(\tau, F) \in \mathfrak{C}_q(\partial\Omega)$ and put $\nu := \mathbb{T}(\tau, F)$. Then $\nu \in \mathbb{M}_q(\partial\Omega)$ if and only if*

$$\tau \in \mathbb{M}_q(\partial\Omega), \quad F = b_q(F) \cup F_\tau. \tag{5.61}$$

(iii) *Let $\nu \in \mathbb{M}_q(\partial\Omega)$ and denote*

$$\begin{aligned} \mathcal{E}_\nu &:= \{E : E \text{ } q\text{-quasi-closed}, \ \nu(E) < \infty\}, \\ \mathcal{D}_\nu &:= \{D : \ D \text{ } q\text{-open}, \ \tilde{D} \overset{q}{\sim} E \text{ for some } E \in \mathcal{E}_q\}. \end{aligned} \tag{5.62}$$

Then a solution of (5.60) *is given by $u = v \oplus U_F$ where*

$$G := \bigcup_{\mathcal{D}_\nu} D, \quad F := \partial\Omega \setminus G, \quad v := \sup\{u_{\nu\chi_E} : \ E \in \mathcal{E}_\nu\}. \tag{5.63}$$

Note that if $E \in \mathcal{E}_\nu$ then $\nu\chi_E$ is a bounded Borel measure which does not charge sets of $C_{2/q,q'}$-capacity zero. Recall that if μ is a positive measure possessing these properties then u_μ denotes the moderate solution with boundary trace μ.

(iv) *The solution $u = v \oplus U_F$ is σ-moderate and it is the unique solution of problem* (5.60) *in the class of σ-moderate solutions. Furthermore, u is the largest solution of the problem.*

PROOF. First we prove

(**A**) *If $u \in \mathcal{U}(\Omega)$*

$$\operatorname{tr} u = \nu \implies \nu \in \mathbb{M}_q(\partial\Omega). \tag{5.64}$$

By Theorem 5.11, u_{reg} is σ-moderate and $u \underset{\mathcal{R}(u)}{\approx} u_{reg}$. Therefore

$$(\operatorname{tr} u)\chi_{\mathcal{R}(u)} = (\operatorname{tr} u_{reg})\chi_{\mathcal{R}(u)}.$$

By Theorem 5.13, $\bar{\mu}_{\mathcal{R}} := \operatorname{tr} u_{reg} \in \mathbb{M}_q(\partial\Omega)$. If v is defined as in (5.63) then

$$v = \sup\{[u]_Q : Q \text{ } q\text{-open regular set}\} = u_{reg}, \tag{5.65}$$

where the second equality holds by definition. Indeed, by Theorem 5.14, for every q-open set Q, $[u]_Q$ is moderate if and only if $\nu(\tilde{Q} \setminus A) < \infty$ for some set A of capacity zero. This means that $[u]_Q$ is moderate if and only if there exists $E \in \mathcal{E}_\nu$ such that $Q \overset{q}{\sim} E$. When this is the case,

$$\operatorname{tr}[u]_Q = \mu_{\mathcal{R}}(u)\chi_{\tilde{Q}} = \mu_{\mathcal{R}}(u)\chi_E = \nu\chi_E.$$

Thus $v \geq u_{reg}$. On the other hand, if $E \in \mathcal{E}_\nu$, then $\tilde{E} \overset{q}{\subset} \mathcal{R}(u)$ and $\mu_{\mathcal{R}}(u)(\tilde{E}) = \mu_{\mathcal{R}}(u)(E) < \infty$. Therefore, by Theorem 5.11 (ix), $\tilde{E}$ is regular, i.e., there exists a q-open regular set Q such that $E \overset{q}{\subset} Q$. Hence $u_{\nu\chi_E} \leq [u]_Q$ and we conclude that $v \leq u_{reg}$. This proves (5.65). In addition, if $E \cap \mathcal{S}(u) \neq \emptyset$ then, by Definition 5.12, $\nu(E) = \infty$. Therefore ν is outer regular with respect to q-topology.

Next we must show that ν is essentially absolutely continuous. Let Q be a q-open set and A a non-empty Borel subset of Q such that $C_{2/q,q'}(A) = 0$. Either

$\nu(Q\setminus A)=\infty$ in which case $\nu(Q\setminus A)=\nu(Q)$ or $\nu(Q\setminus A)<\infty$. In the second case $Q\setminus A\subset\mathcal{R}(u)$ and

$$\nu(Q\setminus A)=\bar\mu_{\mathcal{R}}(Q\setminus A)=\bar\mu_{\mathcal{R}}(Q)<\infty.$$

Let $\xi\in A$ and let D be a q-open subset of Q such that $\xi\subset D\subset\tilde D\subset Q$. Let B_n be a q-open neighborhood of $A\cap\tilde D$ such that $C_{2/q,q'}(B_n)<2^{-n}$ and $B_n\overset{q}{\subset}D$. Then

$$[u]_D\leq[u]_{E_n}+[u]_{B_n},\qquad E_n:=\tilde D\setminus B_n.$$

Since $\lim[u]_{B_n}=0$ it follows that $[u]_D=\lim[u]_{E_n}$. But

$$\|[u]_{E_n}\|_{L^q(\Omega,\rho)}\leq C\nu(E_n)\leq C\nu(Q\setminus A).$$

By assumption, $\nu(Q\setminus A)<\infty$. Therefore $\|[u]_D\|_{L^q(\Omega,\rho)}<\infty$ which implies that $D\subset\mathcal{R}(u)$. Since every point of A has a neighborhood D as above we conclude that $A\subset\mathcal{R}(u)$ and hence $\nu(A)=(\mathrm{tr}_{\mathcal{R}}u)(A)=0$. In conclusion ν is essentially absolutely continuous and $\nu\in\mathbb{M}_q(\partial\Omega)$.

Secondly we prove:

(B) *Suppose that $(\tau,F)\in\mathfrak{C}_q(\partial\Omega)$ satisfies (5.61) and put $\nu:=\mathbb{T}(\tau,F)$. Then the solution $u=v\oplus U_F$, with v as in (5.63), satisfies $\mathrm{tr}\,u=\nu$.*

By (5.64), this also implies that $\nu\in\mathbb{M}_q(\partial\Omega)$.

Clearly v is a σ-moderate solution. The fact that τ is q-locally finite in F^c and essentially absolutely continuous relative to $C_{2/q,q'}$ implies that

$$G:=\partial\Omega\setminus F\subset\mathcal{R}(v),\quad (\mathrm{tr}\,v)\chi_G=\tau\chi_G. \tag{5.66}$$

It follows from the definition of v that $F_\tau\subset\mathcal{S}(v)$. Hence, by Lemma 5.15,

$$F_\tau\cup b_q(F)\subset\mathcal{S}(v)\cup\mathcal{S}(U_F)\subset\mathcal{S}(u)\subset F. \tag{5.67}$$

Hence, by (5.61), $\mathcal{S}(u)=F$, $v=u_{reg}$ and $\tau=\mathrm{tr}\,u_{reg}$. Thus $\mathrm{tr}^c u=(\tau,F)$ which is equivalent to $\mathrm{tr}\,u=\nu$.

Next we show:

(C) *Suppose that $(\tau,F)\in\mathfrak{C}_q(\partial\Omega)$ and that there exists a solution u such that $\mathrm{tr}^c u=(\tau,F)$. Then,*

$$\tau=\mathrm{tr}_{\mathcal{R}}u=\mathrm{tr}\,u_{reg},\quad F=\mathcal{S}(u). \tag{5.68}$$

If $U:=u_{reg}\oplus U_F$ then $\mathrm{tr}\,U=\mathrm{tr}\,u$ and $u\leq U$. U is the unique σ-moderate solution of (5.60) and (τ,F) satisfies condition (5.61).

Assertion (5.68) follows from Theorem 5.14 (i) and Definition 5.12. Since u_{reg} is σ-moderate, it follows, by Theorem 5.13, that $\tau\in\mathbb{M}_q(\partial\Omega)$.

By Theorem 5.11 (vi), $u\underset{\mathcal{R}(u)}{\approx}u_{reg}$. Therefore $w:=u\ominus u_{reg}$ vanishes on $\mathcal{R}(u)$ so that $w\leq U_F$. Note that $u-u_{reg}<w$ and therefore

$$u\leq u_{reg}\oplus w\leq U. \tag{5.69}$$

By their definitions $\mathcal{S}_0(u)=F_\tau$ and by Theorem 5.14 (vi) and Lemma 5.15,

$$\begin{aligned}\mathcal{S}(U)&=\mathcal{S}(u_{reg})\cup\mathcal{S}(U_F)=\mathcal{S}(u_{reg})\cup b_q(F),\\ &=\mathcal{S}_0(u)\cup b_q(F)=F_\tau\cup b_q(F).\end{aligned} \tag{5.70}$$

On the other hand, $\mathcal{R}(U)\supset\mathcal{R}(u_{\mathcal{R}})=\mathcal{R}(u)$ and, as $u\leq U$, $\mathcal{R}(U)\subset\mathcal{R}(u)$. Hence $\mathcal{R}(U)=\mathcal{R}(u)$ and $\mathcal{S}(U)=\mathcal{S}(u)$. Therefore, by (5.68) and (5.70), $F=$

$\mathcal{S}(U) = F_\tau \cup b_q(F)$. Thus (τ, F) satisfies (5.61) and $\operatorname{tr}^c U = (\tau, F)$. The fact that U is the maximal solution with this trace follows from (5.69).

The solution U is σ-moderate because both $u_{\mathcal{R}}$ and U_F are σ-moderate solutions. This fact, with respect to U_F, was proved in [**16**].

The uniqueness of the solution in the class of σ-moderate solutions follows from Theorem 5.13 (iv).

Finally we prove:

(D) *If* $\nu \in \mathbb{M}_q(\partial\Omega)$ *then the couple* (τ, F) *defined by*

$$v := \sup\{u_{\nu\chi_E} : E \in \mathcal{E}_\nu\},\ \tau := \operatorname{tr} v,\ F := \partial\Omega \setminus \mathcal{R}(v) \tag{5.71}$$

satisfies (5.61). *This is the unique couple in* $\mathfrak{C}(\partial\Omega)$ *satisfying* $\nu = \mathbb{T}(\tau, F)$.

The solution v is σ-moderate so that $\tau \in \mathbb{M}_q(\partial\Omega)$.

We claim that $u := v \oplus U_F$ is a solution with boundary trace $\operatorname{tr}^c u = (\tau, F)$. Indeed $u \geq v$ so that $\mathcal{R}(u) \subset \mathcal{R}(v)$. On the other hand, since τ is q-locally finite on $\mathcal{R}(v) = \partial\Omega \setminus F$, it follows that $\mathcal{S}(u) \subset F$. Thus $\mathcal{R}(v) \subset \mathcal{R}(u)$ and we conclude that $\mathcal{R}(v) = \mathcal{R}(u)$ and $F = \mathcal{S}(u)$. This also implies that $v = u_{reg}$.

Finally

$$\mathcal{S}(u) = \mathcal{S}(v) \cup \mathcal{S}(U_F) = F_\tau \cup b_q(F),$$

so that F satisfies (5.61).

The fact that, for $\nu \in \mathbb{M}_q(\partial\Omega)$, the couple (τ, F) defined by (5.71) is the only one in $\mathfrak{C}(\partial\Omega)$ satisfying $\nu = \mathbb{T}(\tau, F)$ follows immediately from the definition of these spaces.

Statements **A**–**D** imply (i)–(iv). □

Remark. If $\nu \in \mathbb{M}_q(\partial\Omega)$ then G and v as defined in (5.63) have the following alternative representation:

$$v := \sup\{u_{\nu\chi_Q} :\ Q \in \mathcal{F}_\nu\}, \quad G = \bigcup_{\mathcal{F}_\nu} Q = \bigcup_{\mathcal{E}_\nu} E, \tag{5.72}$$

$$\mathcal{F}_\nu := \{Q :\ Q \ q\text{-open},\ \nu(Q) < \infty\}. \tag{5.73}$$

To verify this remark we first observe that Lemma 2.6 implies that if A is a q-open set then there exists an increasing sequence of q-quasi closed sets $\{E_n\}$ such that $A = \cup_1^\infty E_n$. In fact, in the notation of (2.22), we may choose $E_n = F_n \setminus L$ where $L = A' \setminus A$ is a set of capacity zero.

Therefore

$$\bigcup_{\mathcal{D}_\nu} D \subset \bigcup_{\mathcal{F}_\nu} Q \subset \bigcup_{\mathcal{E}_\nu} E =: H.$$

On the other hand, if $E \in \mathcal{E}_\nu$ then $\mu_{\mathcal{R}}(u)(\tilde{E}) = \mu_{\mathcal{R}}(u)(E) = \nu(E) < \infty$ and, by Theorem 5.11 (ix), $\tilde{E}$ is regular, i.e., there exists a q-open regular set Q such that $E \overset{q}{\subset} Q$. Thus $H = \bigcup_{\mathcal{D}_\nu} D$.

If D is a q-open regular set then $D = \cup_1^\infty E_n$, where $\{E_n\}$ is an increasing sequence of q-quasi closed sets. Consequently,

$$u_{\nu\chi_D} = \lim u_{\nu\chi_{E_n}}.$$

Therefore

$$\sup\{u_{\nu\chi_Q} :\ Q \in \mathcal{D}_\nu\} \leq \sup\{u_{\nu\chi_Q} :\ Q \in \mathcal{F}_\nu\} \leq \sup\{u_{\nu\chi_E} :\ E \in \mathcal{E}_\nu\}.$$

On the other hand, if $E \in \mathcal{E}_\nu$ then there exists a q-open regular set Q such that $E \overset{q}{\subset} Q$. Consequently we have equality.

Note that, in view of this remark, Theorem 1.3 is an immediate consequence of Theorem 5.16.

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